Generalised Symmetries and Geometric Engineering

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Abstract

In this thesis we introduce the rapidly developing study of generalised global symmetry, a profound expansion of our ordinary notion of symmetry that provides a powerful new way to study Quantum Field Theories (QFTs). We focus our attention to Higher-form Symmetry, or p-form symmetry, in both the continuous and discrete cases, the latter of which being of particular emphasis. We provide examples of higher-form symmetry in well-known theories, as well as some more unfamiliar cases such as BF Theory. A main property of higher-form symmetries that we study in this thesis is their anomalies, particularly the 't Hooft anomalies that arise from the obstruction to gauging these symmetries. After our introduction to generalised symmetry, we turn to the geometric engineering of QFTs from string theory and M-theory, and demonstrate how we can use methods from algebraic topology and homological algebra to determine the generalised symmetries that are present in the resulting QFT. We consider how non-commutativity of the background fluxes in the string theory force us to consider only subgroups of the possible symmetry, and that these choices correspond to selecting global structures for the theory. Finally, we give a brief consideration of how the Symmetry TFT ties together many of the concepts introduced throughout the thesis.

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Contents

Chapter 1

Introduction

In this thesis, we introduce the notion of generalised symmetry, which has seen a great amount of attention since the seminal paper of Gaiotto, Kapustin, Seiberg, and Willett [24]. In particular, we focus on higher-form symmetries, or p-form symmetries, in both the continuous and discrete cases. We emphasise the anomalies of these symmetries, and the defects on which these symmetries act. After our introduction of higher-form symmetries[1], we turn to the geometric engineering of Quantum Field Theories (QFTs), which allows us to use string theories to study the properties of QFTs when gravity decouples. A main aim of the thesis is to show how we can study the higher-form symmetries of QFTs merely by considering the topology of the extra dimensions of the string theory in the geometric engineering setup. In particular, we see how the topology allows us to determine what possible higher-form symmetries are present in the resulting QFT, and by considering the non-commutativity of the string fluxes in the presence of torsional homology, we see how to decide exactly which symmetries are allowed in the theory once we give it a global structure. The plan of the thesis is as follows: In Chapter 1, we introduce the language of ordinary symmetries that allows us to easily progress to generalised symmetry in Chapter 2, as well as introducing the required topics in algebraic topology and algebraic geometry that we will need for our discussions of geometric engineering in Chapter 3. In Chapter 2, we begin our discussion of higher-form symmetries of QFTs , giving illustrative examples as well as important general properties of higher-form symmetries. We introduce continuous higher-form symmetries first, and then move to the less familiar discrete higher-form symmetries. In Chapter 3, we introduce the main ideas and techniques of geometric engineering, and then move on to discussing how we can study higher-form symmetries from a geometric engineering perspective. The final section of Chapter 3 is devoted to flux non-commutativity, where we take quantum effects into account to find which higher-form symmetries can be simultaneously realised in a given QFT. In Chapter 4, we study the

[1]Different types of generalised symmetry were proposed in [24], of which higher-form symmetries are the most familiar to those unacquainted with generalised symmetry, which is why we focus only on these in this thesis.

anomalies of gauge theories and their connection to global structures, as well as giving an introduction to Symmetry TFTs. We assume a mathematical knowledge of differential forms, differential geometry, and basic abstract algebra (such as groups and rings), and a physics knowledge of supersymmetric and conformal quantum field theories, and bosonic string theory, to the level of an advanced Masters' course[2].

1.1 Ordinary Symmetries

The aim of this section is to briefly introduce some notation and conventions regarding our ordinary notion of symmetry in a way that will make it simple to introduce generalised symmetry. We will use differential form notation to simplify our discussions.

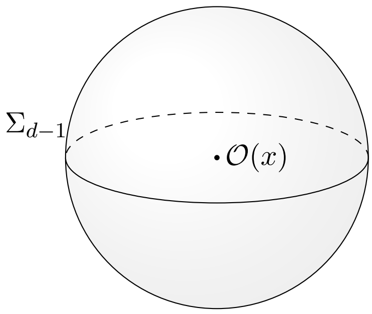
When we have a global symmetry in physics, by Noether's theorem we know that there is a conserved (1-form) current of the form

$$j_\mu \leftrightarrow j_1 \text{ such that } \; \partial_\mu j^\mu = 0 \leftrightarrow d * j_1 = 0 \tag{1.1}$$

This 1-form current arises from what is called, in the language of generalised symmetry, a 0-form symmetry, i.e. just our usual concept of symmetry. From this current we can obtain a conserved charge operator within a $(d-1)$-dimensional surface Σ_{d-1}

$$Q(\Sigma_1) = \int_{\Sigma_{d-1}} *j_1 \tag{1.2}$$

which is the generator of our symmetry. A visual way of representing this is the following: let $\mathcal{O}(x)$ be some operator that is charged under our symmetry, then $Q(\Sigma_{d-1})$ measures the total charge inside the surface Σ_{d-1}:



If we exponentiate the charge operator we get our symmetry operator

$$\mathcal{U}_g(\Sigma_1) = e^{i\alpha \int_{\Sigma_{d-1}} *j_1} \tag{1.3}$$

[2]Such as the MSc in Particles, Strings, and Cosmology, perhaps.

where $g = e^{i\alpha} \in G$, the symmetry group [13]. Here, G could be some potentially non-abelian group, with $\alpha = \alpha^j T^j$. It is known already for 0-form symmetries that, so long as we don't cross operator insertions, we can smoothly deform Σ_{d-1} and so these symmetry operators are *topological*.

We will see in the next chapter how we can introduce the notion of p-form symmetries, and that they extend very naturally from the language of symmetry in this section.

1.2 Relative Homology, Tor and Ext

For our discussion of geometric engineering, we will need to introduce some concepts from algebraic topology. In particular, the homology of non-compact spaces will be crucial, as well as homology groups with torsion.

Given a collection of abelian groups A_n and homomorphisms $h_n : A_n \to A_{n-1}$, we can write a **sequence** as

$$\ldots \to A_{n+1} \xrightarrow{h_{n+1}} A_n \xrightarrow{h_n} A_{n-1} \to \ldots \tag{1.4}$$

If we have that $Ker(h_n) = Im(h_{n+1})$ for all h_n then we say that this is a **(long) exact sequence**. If we have an exact sequence of the form

$$0 \to A \xrightarrow{h} B \xrightarrow{g} C \to 0 \tag{1.5}$$

where h is injective and g is surjective, then we call this a **short exact sequence** [29]. A useful property of short exact sequences, or more generally the end of a long exact sequence, is that we have

$$C \cong B/A \tag{1.6}$$

as if h allows us to view A as a subgroup of B, but any $b \in B$ s.t $b = h(a)$ for $a \in A$ will have $g(b) = 0$, while for b that cannot be written in this way we have a mapping to non-trivial elements of C. Another useful exact sequence is

$$0 \to A \to B \to 0 \tag{1.7}$$

where we have that this is exact iff $A \cong B$ [29].

If we have an exact sequence with $Im(h_{n+1}) \subset Ker(h_n)$, then we have $h_n \circ h_{n+1} = 0$, which forms a **chain complex** [29]. This naturally allows us to define the $\mathbf{n^{th}}$ **homology group** of the chain complex

$$H_n = Ker(h_n)/Im(h_{n+1}) \tag{1.8}$$

Of interest to us is the homology of manifolds, and so we define $\mathcal{C}_k$ to be the k-dimensional submanifolds, **k-chains**, of an orientable, closed d-dimensional manifold $\mathcal{M}_d$, and boundary maps $\partial_{k+1} : \mathcal{C}_{k+1} \to \mathcal{C}_k$ with the property $\partial_k \circ \partial_{k+1} = 0$. If $\partial_k S_k = 0$ for $S_k \in \mathcal{C}_k$, we call S_k a **k-cycle**. Then, we can define the k^{th} homology group of $\mathcal{M}_d$ as

$$H_k(\mathcal{M}_d, \mathbb{Z}) = Ker(\partial_k)/Im(\partial_{k+1}) \tag{1.9}$$

where the $\mathbb{Z}$ in this definition denotes that the homology group is defined with additive group operation, with integer coefficients [44]. Said plainly, the k^{th} homology group is the group of k-dimensional submanifolds without boundary, that are not the boundary of some $(k+1)$-dimensional submanifold.

We note some useful properties of the homology groups now [44]:

- A class $[S_k] \in H_k(\mathcal{M}_d, \mathbb{Z})$ is the space of submanifolds that differ by a boundary, i.e. $S_k \sim S_k + \partial_{k+1} S_{k+1}$
- We define the Betti numbers $b_k = dim H_k(\mathcal{M}_d, \mathbb{R})$. We have that the Euler Characteristic is $\chi(\mathcal{M}_d) = \Sigma_{k=0}^{d} (-1)^k b_k$.
- $H_0(\mathcal{M}_d, \mathbb{Z}) = \mathbb{Z}, \; H_d(\mathcal{M}_d, \mathbb{Z}) = \mathbb{Z}$
- The **Künneth formula** relates the homology groups of product manifolds $M \times M'$, where we assume that M is torsion-free for simplicity (as we always assume that this is the case in this thesis):

$$H_k(M \times M', \mathbb{Z}) = \bigoplus_{i+j=k} H_i(M, \mathbb{Z}) \otimes H_j(M', \mathbb{Z}) \tag{1.10}$$

We can also define the **de Rham cohomology groups** as the differential forms that are closed but not exact [40]:

$$H^k(\mathcal{M}_d) = Ker(d_k)/Im(d_{k+1}) \tag{1.11}$$

where we denote in a verbose way d_k as the exterior derivative acting on k-forms. Note that we use a superscript for cohomology and a subscript for homology.

A useful connection between the de Rham cohomology and homology exists, known as **Poincaré Duality** [44]:

$$H^k(\mathcal{M}_d) \cong H_{d-k}(\mathcal{M}_d, \mathbb{Z}) \tag{1.12}$$

which allows us to take integration of a k-form on a k-cycle to integration on the entire manifold; let $PD[\alpha_k] \equiv S_{d-k}$ where $\alpha_k \in H^k(\mathcal{M}_d)$ is a k-form with Poincaré dual $S_{d-k} \in H_{d-k}(\mathcal{M}_d, \mathbb{Z})$, such that

$$\int_{S_{d-k}} \beta_{d-k} = \int_{\mathcal{M}_d} \alpha_k \wedge \beta_{d-k} \tag{1.13}$$

for all $(d-k)$-forms $\beta_{d-k} \in H^{d-k}(\mathcal{M}_d)$.

We now move onto the two most important aspects of homology for this thesis: Relative Homology, and Torsion. We can decompose

$$H_k(\mathcal{M}_d, \mathbb{Z}) = \mathbb{Z}^{b_k} \oplus_p \mathbb{Z}_p \tag{1.14}$$

where the sum of $\mathbb{Z}_p$'s is called the **torsion** [44]. We usually denote this as $Tor H_k(\mathcal{M}_d, \mathbb{Z})$, but this is not to be confused with the Tor functor which we introduce shortly. Not all manifolds exhibit torsion, but we will see later that torsional homology in geometric engineering setups leads to discrete p-form symmetries in QFTs and so we will necessarily need to include torsion in our geometric engineering setups if we wish to see these symmetries arise. If we are considering torsion in cohomology, it is slightly different; let's denote $H_k(\mathcal{M}_d, \mathbb{Z}) = \mathbb{Z}^{b_k} \oplus T_k$, where T_k is the torsion. Then we have

$$H^k(\mathcal{M}_d) = \mathbb{Z}^{b_k} \oplus T_{k-1} \tag{1.15}$$

i.e., the non-torsional sector of the homology and cohomology groups are the same, but not the torsional sector [44].

We will sometimes write all of the homology groups in a condensed way

$$H_\bullet(\mathcal{M}_d, \mathbb{Z}) = \{H_0(\mathcal{M}_d, \mathbb{Z}), ..., H_d(\mathcal{M}_d, \mathbb{Z})\} \tag{1.16}$$

and similar for cohomology.

Now we discuss relative homology. In geometric engineering, we wish to consider string theories or M-Theory with non-compact extra dimensions, and so we need to equip ourselves with a notion of homology that applies to these spaces. Let $\mathcal{M}_d$ be an orientable, non-compact manifold with boundary $\partial\mathcal{M}_d$. Then we can loosen our notion of homology slightly, to allow for k-cycles $S_k \in \mathcal{C}_k$ such that $\partial_k S_k \subset \partial\mathcal{M}_d$ instead of the stricter condition $\partial_k S_k = 0$. We call these **relative k-cycles**: k-chains whose boundary lies only along the boundary of the manifold, and denote the space of relative k-cycles as $\mathcal{C}_k^\partial$. Note that our ordinary notion of k-cycles are also relative k-cycles; their boundary is the empty set, which is technically contained within the set of points of the boundary $\partial\mathcal{M}_d$. Then, we define the $k^{\mathbf{th}}$ **relative homology group** as [44]

$$H_k(\mathcal{M}_d, \partial\mathcal{M}_d) = \mathcal{C}_k^\partial / Im\partial_{k+1} \tag{1.17}$$

which has the properties [44]

$$H_k(\mathcal{M}_d, \mathbb{Z}) \subset H_k(\mathcal{M}_d, \partial\mathcal{M}_d) \tag{1.18}$$

$$H_0(\mathcal{M}_d, \partial\mathcal{M}_d) = 0 \tag{1.19}$$

There exists an exact sequence [29]

$$\ldots \to H_k(\partial\mathcal{M}_d, \mathbb{Z}) \to H_k(\mathcal{M}_d, \mathbb{Z}) \to H_k(\mathcal{M}_d, \partial\mathcal{M}_d) \to H_{k-1}(\partial\mathcal{M}_d, \mathbb{Z}) \to \ldots \tag{1.20}$$

where we have suppressed the particular homomorphisms that give this sequence. We can see that this is iterative; the ellipses on the right will be the same sequence, just with k reduced by 1. Thus, if we assume that the sequence terminates at the end i.e., $H_{k-1}(\mathcal{M}_d) = 0$, then we can use the following property, using Equation 1.6

$$H_{k-1}(\partial\mathcal{M}_d, \mathbb{Z}) = H_k(\mathcal{M}_d, \partial\mathcal{M}_d)/H_k(\mathcal{M}_d, \mathbb{Z}) \tag{1.21}$$

This equation will end up being of great importance to us later, allowing us to simplify homological computations in the geometric engineering of higher-form symmetries.

We now introduce some useful functors from homological algebra that will be helpful for us in this thesis, namely Tor and Ext. These are called **derived functors**, for reasons we won't explain in this thesis. In fact, our discussion of Tor and Ext will illuminate very little of the purpose of these functors in homological algebra, but we simply need them as a tool to utilise some theorems that are of use to us. A proper introduction to these functors, and homological algebra in general, can be found in [46, 39].

Instead of stating what these two functors are by definition, we will simply give their properties for abelian groups. First, we have the following properties of **Tor** functor for (non-trivial) abelian groups A, B [46]:

- A is torsion free $\Leftrightarrow Tor(A, B) = 0 \Leftrightarrow Tor(B, A) = 0$
- $Tor(\mathbb{Z}_n, B) =_n B = \{b \in B | nb = 0\}$
- $Tor(\mathbb{Z}^b \oplus_i \mathbb{Z}_{n_i}, B) = \oplus_i \, Tor(\mathbb{Z}_{n_i}, B)$

We note that what we are calling Tor should actually be written as $Tor_1^{\mathbb{Z}}$, and that this is just one of the possible Tor functors, but this is all we will need in this thesis.

Next, we state some properties of the **Ext** functor for abelian groups A, B, as well as a useful application to short exact sequences [39]:

$$Ext(\mathbb{Z}_n, B) = \frac{B}{nB} \tag{1.22}$$

$$Ext(\mathbb{Z}, B) = 0 \tag{1.23}$$

and then if we have a short exact sequence

$$0 \to B \to C \to A \to 0 \tag{1.24}$$

then $C = A \oplus B$, i.e. the sequence **splits**, if $Ext(A, B) = 0$ [46]. If $Ext(A, B) \neq 0$, then we have that this is measuring all of the possible 'equivalence classes'[3] of possible C, with the zero element of $Ext(A, B)$ being the split sequence [46]. Again, what we are calling Ext should actually be written as $Ext^1_{\mathbb{Z}}$, but as before we will only need this functor instead of other possible Ext functors and so refer to it as Ext.

1.3 Toric Varieties

As well as the algebraic topology in the previous section, we must also give an introduction to some algebraic geometry. The previous section introduced relative homology and torsion of a general non-compact manifold with boundary, $\mathcal{M}_d$, and the aim of this section is to introduce the language we need to give examples of $\mathcal{M}_d$. We will be considering singular toric Calabi-Yau 3-folds, and the methods used to extract the homological data from these 3-folds using toric diagrams.

Denote the **polynomial ring over a field** $\mathbb{F}$ as $\mathbb{F}[z_1, ..., z_n]$. This is just the ring of polynomials in variables $z_1, ..., z_n$ with coefficients in $\mathbb{F}$. We will quickly specify our field to be $\mathbb{C}$, as this is the only field we will be considering in this thesis. We can then define an n-dimensional **ideal of polynomials of S** for $S \subseteq \mathbb{C}^n$ as the following [28]

$$I(S) = \{f \in \mathbb{C}[z_1, ..., z_n] \mid f(s) = 0 \ \forall s \in S\} \tag{1.25}$$

Put simply, $I(S)$ is just the set of all polynomials that are zero for all points in some subset S of the complex plane $\mathbb{C}^n$. It is clear to see that $I(\mathbb{C}^n) = \{0\}$, as no non-trivial polynomial can be zero at every point in the entire complex plane [28]. Another simple example is $S = \{(z_1, 0) \mid z_1 \in \mathbb{C}\}$, which has

$$I(S) = \mathbb{C}[z_2] \tag{1.26}$$

Next, we can define an **affine variety** of an ideal $I \subset \mathbb{C}[z_1, ..., z_n]$ as [17]

$$V(I) = \{s \in \mathbb{C}^n \mid f(s) = 0 \ \forall f \in I\} \tag{1.27}$$

We do not have, however, that $V(I(S)) = S$. Using an example $I(S)$ from [28], let $S = \{z \in \mathbb{C} \mid |z|^2 = 1, \ Re(z) \neq 0\}$, i.e. a circle without the north and sound pole. Then, the ideal for S is

$$I(S) = \langle |z|^2 - 1 \rangle \tag{1.28}$$

i.e. the polynomials in z generated by the term in the angular brackets. If we take this ideal, then the corresponding affine variety is

$$V(I(S)) = \{z \in \mathbb{C} \mid |z|^2 = 1\} \tag{1.29}$$

[3] See [46] for more on group extensions and Ext.

which is now the whole circle, including the north and south poles. From this we can see that $V(I(S)) \neq S$ in general.

Now we are in a position to begin defining some toric varieties. We define the n-dimensional **algebraic torus** $(\mathbb{C}^*)^n$ as [17]

$$(\mathbb{C}^*)^n := \mathbb{C}^n \setminus V(\{z_1...z_n\}) \tag{1.30}$$

where the affine variety V is essentially removing all of the coordinate axes from the complex n-plane such that we are left only with the 'quadrants'.

Then, we can define an n-dimensional **toric variety** as [41]

$$X_\Sigma = (\mathbb{C}^N \backslash F_\Sigma)/(C^*)^m \tag{1.31}$$

where $m < N, n = N - m$. The m-dim algebraic torus acts by coordinate multiplication, and F_Σ are the set of points left invariant by the torus action, and thus must be removed.

We can encode the variety X_Σ by a lattice N isomorphic to $\mathbb{Z}^n$ and its fan Σ. Before we discuss this any further, we must explain what we mean by a fan.

Firstly, a **k-dimensional strongly convex rational polyhedral cone** $\sigma \subset N \otimes \mathbb{R}$ is a set [32]

$$\sigma = \{a_1 v_1 + ... + a_k v_k \mid a_i \geq 0,\ a_i \in \mathbb{R}\} \tag{1.32}$$

such that $\sigma \cap (-\sigma) = \{0\}$, where we say that σ is generated by the vectors $\{v_i\}$. The generating vectors have integer coordinates, and span a subspace of the lattice N. We will refer to r-dimensional subsets of σ as **r-dimensional faces** of the cone generated by some r-dimensional subset of the generating vectors of σ, such that the faces are also r-dimensional cones. An intersection of two faces is also a face of σ, as is the face of a face [17].

Now, we define a **fan** Σ as a collection of strongly convex rational polyhedral cones such that each face of a cone in Σ is also in Σ, and the intersection of two cones in Σ is a face of each cone [32]. We can write the fan Σ in terms of the generators of the one-dimensional cones $\{v_i\}$, of which there are n, and the d-dimensional cones in Σ correspond to codimension-d cycles in X_Σ [41].

As mentioned at the beginning of this section, we are interested in singular toric Calabi-Yau 3-folds, which means we are looking for toric varieties with 3 generating vectors v_i. To ensure that our varieties are **Calabi-Yau**, this corresponds to having these v_i all lying in the same hyperplane, one unit away from the origin of the lattice N, and so we can set, for instance, the third component of each v_i to 1:

$$(v_i)_3 = 1 \tag{1.33}$$

This means that we can identify the toric Calabi-Yau in question by drawing the v_i vectors in a plane [41].
Now, we will step away from toric varieties for just a second to discuss orbifolds. We define an **orbifold** as the quotient of a manifold M by a discrete group Γ [41]:

$$X = M/\Gamma \tag{1.34}$$

Of interest to us are the orbifolds with $M = \mathbb{C}^n$ and $\Gamma = \mathbb{Z}_p$. To be clear, we intend to use orbifolds exactly of this kind as our extra dimensions in our geometric engineering setups. For $n = 3$, we will show that orbifolds of this type happen to be singular toric Calabi-Yau 3-folds.

A rational polyhedral cone is **simplicial** if its generating vectors v_i form a basis for the vector space that they span, i.e. if they are linearly independent. A fan Σ is simplicial if each cone in Σ is simplicial. A toric variety is an orbifold iff its fan is simplicial [32]. Therefore, to ensure that we obtain an orbifold, we require the generating v_i of our toric variety X_Σ above to be linearly independent such that all the faces in Σ will also be simplicial, thus giving that Σ is simplicial and X_Σ is an orbifold. We will show now how picking different linearly independent v_i corresponds to picking different orbifolds $X_\Sigma = \mathbb{C}^3/\mathbb{Z}_p$.

On X_Σ we can pick local coordinates

$$u^i = z_1^{(v_1)_i} z_2^{(v_2)_i} z_3^{(v_3)_i} \tag{1.35}$$

where the action of the group $(\mathbb{C}^*)^m$ on the v_i is given by

$$\sum_{i=1}^{3} l_i^{(a)} v_i, \ a = 1, ..., m, \ l_i^{(a)} \in \mathbb{Z} \tag{1.36}$$

such that u_i is invariant under this action [41].

Then, we consider how the discrete group of an orbifold acts on its coordinates. Generally, for discrete group $\mathbb{Z}_p$, we have the coordinates transform as

$$(z_1, z_2, z_3) \to (\epsilon^{n_1} z_1, \epsilon^{n_2} z_2, \epsilon^{n_3} z_3) \tag{1.37}$$

where $\epsilon = e^{\frac{2\pi i}{p}} \in \mathbb{Z}_p$ such that $\epsilon^p = 1$ [41]. The values of n_i must obey specific relations to ensure that the resulting orbifold is Calabi-Yau, this being [41]

$$n_1 + n_2 + n_3 = 0 \bmod p \tag{1.38}$$

We should note that this does not necessarily give a unique n_i triple - in fact, $\mathbb{Z}_6, \mathbb{Z}_8, \mathbb{Z}_{12}$ each have two different ways of acting, and a full table of the allowed $\mathbb{Z}_p$ and their corresponding n_i triple can be found in [41]. For us, we will simply quote these n_i triples as we need them. By convention, one sets $n_1 = 1$ such that we only need to find n_2, n_3.

To see how we can find which orbifold X_Σ corresponds to, we can consider how they act on the u^i coordinates by acting on the z_i coordinates with these orbifold actions and insist that they are invariant [41]

$$u^i \to \epsilon^{(v_1)_i}\epsilon^{n_2(v_2)_i}\epsilon^{n_3(v_3)_i}u^i \stackrel{!}{=} u^i \Rightarrow (v_1)_i + n_2(v_2)_i + n_3(v_3)_i = 0 \bmod p \tag{1.39}$$

Thus, with the Calabi-Yau condition also setting $(v_j)_3 = 1 \;\forall\; j$, we have that for a given $\mathbb{Z}_p$ (and thus fixed values of n_i), all we must do to ensure that X_Σ is the $\mathbb{C}^3/\mathbb{Z}_p$ orbifold is to solve these two remaining equations for linearly independent v_i, which gives us a simplicial fan Σ.

Once we have fixed v_i, we can draw the **toric diagram** of the orbifold by drawing these vectors on a 2-dimensional integer lattice.

As an example, consider $\mathbb{C}^3/\mathbb{Z}_3$. From [41], we have that $(n_1, n_2, n_3) = (1, 1, -2)$, and so we must solve the equations

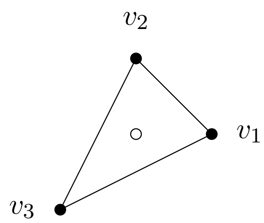
$$(v_1)_1 + (v_2)_1 - 2(v_3)_1 = 0 \bmod 3 \tag{1.40}$$

$$(v_1)_2 + (v_2)_2 - 2(v_3)_2 = 0 \bmod 3 \tag{1.41}$$

such that v_i are all linearly independent. One possible way of writing this is

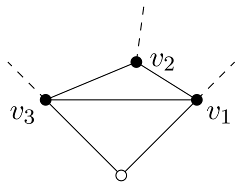
$$v_1 = \begin{bmatrix}1\\0\\1\end{bmatrix}, \; v_2 = \begin{bmatrix}0\\1\\1\end{bmatrix}, \; v_3 = \begin{bmatrix}-1\\-1\\1\end{bmatrix} \tag{1.42}$$

as we can see that these are linearly independent and satisfy the above equations. Drawing these vectors in the z_1, z_2 hyperplane, along with the origin point gives the toric diagram of $\mathbb{C}^3/\mathbb{Z}_3$:



where the point in the middle of this diagram is the origin. As the v_i are 1-dimensional cones, we can make three 2-dimensional cones by connecting these points together (the lines in the toric diagram). The 3-dimensional cone is then given by connecting these three lines together. We can see that $\mathbb{C}^3/\mathbb{Z}^3$ actually looks like a cone[4] by visualising what this would look like in 3-dimensions:

[4]Rather, an upside down, infinitely tall tetrahedron.



where the dotted lines are showing us that this cone extends off to infinity.

In general, we can compute the homology groups from the toric diagram of our $\mathbb{C}^3/\mathbb{Z}_p$ orbifolds. However, as the varieties we have considered so far are singular (due to the singularity at the tip of the cone), one would need to be careful about how to define homology on such spaces. Something we can do instead is to find a **crepant resolution** $X_{\tilde{\Sigma}}$ that smooths the space such that we can talk about its homology without issue. We won't discuss the specifics of how we do this, as the homological data for the crepant resolution is still obtained by the initial toric diagram, but the idea is this: given the toric diagram, we triangulate the diagram by connecting interior points to exterior points, and this corresponds to some smoothed version of the orbifold. Again, this isn't something we intend to do explicitly, but is something we must take into account - in our geometric engineering setups we will be taking some crepant resolution of our singular variety to be the extra dimensions, but the important information that we need for our constructions comes from the toric diagram only.

To obtain the homology our variety, let $\mathcal{I}$ be the number of points inside the toric diagram corresponding to Σ (the fan of the *singular* toric variety), and $\mathcal{B}$ the number of points on the edge of the diagram. Then we have that $\mathcal{I}$ counts the number of complex codimension-1 cycles (i.e., real dimension 4 cycles), and it can be shown that the number of 2-cycles is $\mathcal{I} + \mathcal{B} - 3$, and we have that there are no non-trivial odd-dimensional cycles for toric varieties [3]. Note that this means for toric varieties we can use Equation 1.21. Therefore, we can write the homology groups of the crepant resolution of the variety $X_{\tilde{\Sigma}}$ as

$$H_\bullet(X_{\tilde{\Sigma}}, \mathbb{Z}) = \{\mathbb{Z}, 0, \mathbb{Z}^{\mathcal{I}+\mathcal{B}-3}, 0, \mathbb{Z}^{\mathcal{I}}, 0, 0\} \tag{1.43}$$

For example, if we consider the toric diagram of $\mathbb{C}^3/\mathbb{Z}_3$ that we gave above, then we have $\mathcal{I} = 1, \mathcal{B} = 3$, so the homology of the crepant resolution, call it $\widetilde{\mathbb{C}^3/\mathbb{Z}_3}$, is then

$$H_\bullet(\widetilde{\mathbb{C}^3/\mathbb{Z}_3}, \mathbb{Z}) = \{\mathbb{Z}, 0, \mathbb{Z}, 0, \mathbb{Z}, 0, 0\} \tag{1.44}$$

Currently, away from the singularity this manifold is locally of the form $\mathbb{C}^3$, i.e. just flat space, and thus has no boundary. To provide this space with a boundary, we can simply consider placing a 6-dimensional ball at infinity of the crepant resolution, and call this manifold with boundary X_6. This will give us a boundary of the form [3]

$$\partial X_6 = S^5/\mathbb{Z}_p \tag{1.45}$$

Chapter **2**

Higher-Form Symmetry

In this chapter we aim to give a detailed introduction to higher-form symmetries, sometimes called p-form global symmetries. The first section will introduce the general properties of these symmetries in the continuous case, as well as giving some examples. We will also introduce discrete higher-form symmetries, which will perhaps feel less familiar than their continuous counterpart, but are thus a more interesting case to consider. In particular, it has been predicted that the standard model exhibits such discrete symmetries, which leads to some interesting predictions such as the existence of fractionally charged hadrons [5]. Another main focus for us is to shed light on what these symmetries act on - a 0-form symmetry acts on 0-dimensional operators, i.e. points. So, a p-form symmetry acts on p-dimensional operators, which we call **defects**. Some standard references for introductions to higher-form symmetries are [13, 10, 42].

2.1 Continuous Higher-form Symmetries

In Section 1.1, we considered how a 0-form symmetry gives rise to a 1-form conserved current, and how we were able to create a symmetry operator from the associated conserved charge. This charge was defined over a $(d-1)$-dimensional surface, and could thus surround a 0-dimensional operator.

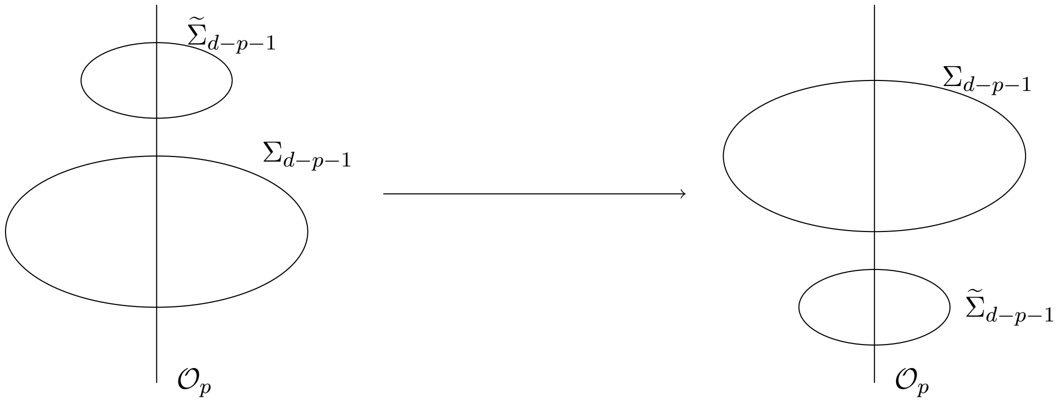
Now suppose that we have some conserved $(p+1)$-form current j_{p+1} such that $d{*}j_{p+1} = 0$. What would this mean for us? Well, in the same way a conserved 1-form current gives us a 0-form symmetry, a conserved $(p+1)$-form current would imply the existence of some p-form symmetry. Consider then what the associated charge operator would be:

$$Q(\Sigma_{d-p-1}) = \int_{\Sigma_{d-p-1}} {*}j_{p+1} \tag{2.1}$$

where Σ_{d-p-1} is some $(d-p-1)$-dimensional surface, and from this we could construct a symmetry operator

$$\mathcal{U}_g(\Sigma_{d-p-1}) = e^{i\alpha \int_{\Sigma_{d-p-1}} *j_{p+1}} \tag{2.2}$$

where again $g = e^{i\alpha} \in G^{(p)}$ is the symmetry group. Notice now that we are referring to the p-form symmetry group as $G^{(p)}$[5]. In Section 1.1, we said that the 0-form symmetry group could potentially be non-abelian, but for higher-form symmetries, $G^{(p)}$ must always be abelian. In [13] there is a way of visualising this by considering a diagram of the following form:



Recall that a 0-form symmetry acts on 0-dimensional operators, but a p-form symmetry acts on p-dimensional operators called defects, e.g. for p=1 we have that the 1-form symmetry acts on lines. The dimensionality of the defects and the topological nature of the symmetry operators means that the surfaces Σ_{d-p-1}, $\widetilde{\Sigma}_{d-p-1}$ have room to exchange positions without crossing each other, and thus the symmetry operators commute. Thus, $G^{(p)}$ is always abelian, and therefore [3]

$$G^{(p)} \subseteq U(1)^N \tag{2.3}$$

for some N . So the fact that $G^{(p)}$ is always abelian means we have

$$\mathcal{U}_g\, \mathcal{U}_{g'} = \mathcal{U}_{gg'} = \mathcal{U}_{g'g} = \mathcal{U}_{g'}\, \mathcal{U}_g \tag{2.4}$$

or explicitly,

$$e^{i\alpha Q(\Sigma)} e^{i\alpha' Q(\Sigma)} = e^{i(\alpha+\alpha')Q(\Sigma)} \tag{2.5}$$

as expected for an abelian symmetry. It's clear to see that setting $g' = g^{-1}$ in the equations above would result in the identity operator on the right-hand side, and so we see that $G^{(p)}$ is in fact a group[6].

[5] This is sometimes referred to as Córdova-Dumitrescu-Intriligator notation.

[6] We note that $G^{(p)}$ can be some more general algebraic structure without inverse, i.e. when $\nexists\, g' = g^{-1}$. These are called non-invertible symmetries, and we will not discuss these in this thesis, but see [43, 42, 13].

We have constructed our symmetry operators, but we have not discussed the defects yet. As mentioned, these will be p-dimensional operators with a charge under the symmetry, which we call q. They are defined by this charge and the p-dimensional submanifold that they lie along:

$$\textbf{Defect} : \mathcal{D}_q(\gamma_p) \tag{2.6}$$

Their precise form is dependent on the theory, and so we will consider an example to shed light on these defects. Our first example will be pure 4d U(1) gauge theory. First, we should discuss how the symmetry operators act on these defects. We have the following Ward identity [13]

$$d * j_{p+1}(x)\mathcal{D}_q(\gamma_p) = q\delta^{(d-p)}(x \in \gamma_p)\mathcal{D}_q(\gamma_p) \tag{2.7}$$

where we define

$$\delta^{(d-p)}(x \in \gamma_p) = \begin{cases} 1 & x \in \gamma_p \\ 0 & x \notin \gamma_p \end{cases} \tag{2.8}$$

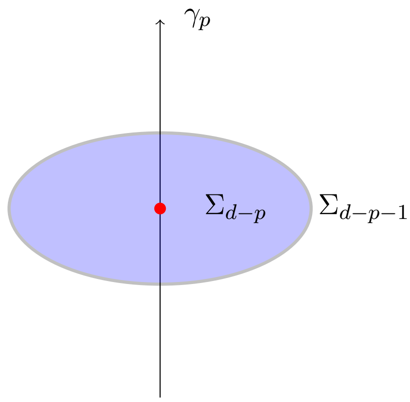
Then, consider multiplying either side of the Ward identity by some $U(1)$ parameter α, taking an integral over some $(d-p)$-dimensional Σ_{d-p} of the action of the Ward identity on the defect and exponentiating:

$$e^{i\alpha \int_{\Sigma_{d-p}} d*j_{p+1}}\mathcal{D}_q(\gamma_p) = e^{i\alpha q \int_{\Sigma_{d-p}} \delta^{(d-p)}}\mathcal{D}_q(\gamma_p) \tag{2.9}$$

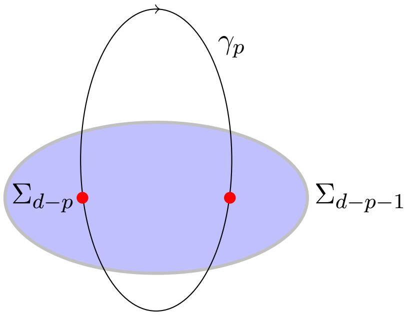
We should consider both of these exponentials independently, and we will see an interesting relationship between them afterwards. For the left hand side, we can let $\partial\Sigma_{d-p} = \Sigma_{d-p-1}$, and then use Stoke's theorem to obtain

$$e^{i\alpha \int_{\Sigma_{d-p}} d*j_{p+1}} = e^{i\alpha \int_{\Sigma_{d-p-1}} *j_{p+1}} = \mathcal{U}_g(\Sigma_{d-p-1}) \tag{2.10}$$

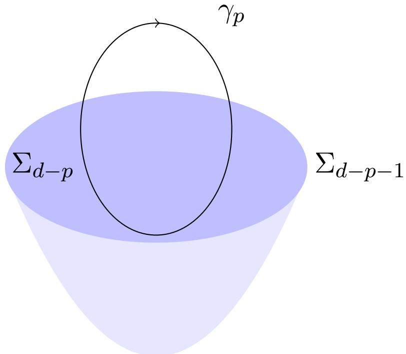
so the left hand side corresponds to the action of the symmetry operator $\mathcal{U}_g(\Sigma_{d-p-1})$ acting on the defect. Then, we can consider the exponential on the right hand side. The main thing we wish to consider is what the integral is equivalent to. Once again, some diagrams are helpful to understand physically what this integral represents. Again, let $\partial\Sigma_{d-p} = \Sigma_{d-p-1}$, and fix some orientation for γ_p [13]



so we can see that integrating this delta function over Σ_{d-p} will be zero everywhere other than at the red dot of the diagram - this integral will thus be equal to 1. Alternatively, we could consider another γ_p



where here there would be two points in the integral that would be non-zero, but the orientation gives them opposite sign and thus they cancel, such that the integral is zero. There is a reason for this - consider the following digram, where we have topologically moved[7] Σ_{d-p} down below γ_p such that they no longer intersect



and there are no intersections at all. What we see is that adding an orientation to γ_p in this integral tells us whether or not Σ_{d-p-1} and γ_p 'link' each other, i.e. whether or not there is some way we can topologically deform Σ_{d-p-1} such that there is no intersection. There may, of course, be more than one intersection as well, in which case this integral will measure this too. We call this integral the **linking number** $\mathrm{Link}(\Sigma_{d-p-1}, \gamma_p)$ [13]. Another way we can write this linking number is the following [14]

$$Link(\Sigma_{d-p-1}, \gamma_p) = \int PD[\Sigma_{d-p}] \wedge PD[\gamma_p] \tag{2.11}$$

and it is sometimes referred to as the intersection number in this form.

Having considered both exponentials, we can write the action of the symmetry operator on the defect as

$$\mathcal{U}_g(\Sigma_{d-p-1})\mathcal{D}_q(\gamma_p) = e^{i\alpha q \mathrm{Link}(\Sigma_{d-p-1},\gamma_p)}\mathcal{D}_q(\gamma_p) \tag{2.12}$$

[7] Technically speaking, we transformed it to a homotopically equivalent surface.

Now that we have discussed both the p-form symmetry operators and defects, we can turn to an example.

4d Maxwell Theory

Consider the usual 4d Maxwell action

$$S = \int_{\mathcal{M}_4} F_2 \wedge *F_2 \tag{2.13}$$

where $F_2 = dA_1$, such that $d^2 = 0$ gives $dF_2 = 0$, the Bianchi Identity. By varying the action, we get the equations of motion:

$$\delta S = 0 = \int d(\delta A_1) \wedge *F_2 \tag{2.14}$$

$$= -\int \delta A_1 \wedge (d * F_2) \tag{2.15}$$

$$\Rightarrow d * F_2 = 0 \tag{2.16}$$

The equations of motion and the Bianchi identity look remarkably like the conservation of a pair of 2-form currents:

$$j_2^{(e)} = F_2,\ j_2^{(m)} = *F_2 \tag{2.17}$$

where the superscript is understood to mean electric current and magnetic current. Applying our newfound understanding of p-form symmetries, we can see that this implies the existence of a pair of 1-form symmetries. We can let $*F_2 = d\tilde{A}_1$ (i.e. we do not assume that F_2 is self dual). $\tilde{A}_1$ is sometimes referred to as the dual photon. Note that this is a particular restriction, and that we don't have to consider $*F_2$ in this way - however, it allows us to see clearly what our symmetries are. A natural way to see these symmetries is by the shifts

$$A_1 \to A_1 + \lambda_1,\ \tilde{A}_1 \to \tilde{A}_1 + \tilde{\lambda}_1 \tag{2.18}$$

where $\lambda_1, \tilde{\lambda}_1$ are two constant 1-forms. These shifts leave the action invariant, and so they are clearly 1-form symmetries. The symmetry of A_1 is sometimes referred to as the **1-form electric symmetry**, and for $\tilde{A}_1$ the **1-form magnetic symmetry**. The corresponding symmetry operators are

$$\mathcal{U}_g^{(e)}(\Sigma_2) = e^{i\alpha \int_{\Sigma_2} *F_2} \tag{2.19}$$

$$\mathcal{U}_g^{(m)}(\tilde{\Sigma}_2) = e^{i\alpha \int_{\tilde{\Sigma}_2} F_2} \tag{2.20}$$

and so all that is left to do is work out what the defects are. For our Maxwell theory, these defects are [13]

$$\mathcal{D}_q^{(e)}(\gamma_1) = e^{iq \int_{\gamma_1} A_1} \tag{2.21}$$

$$\mathcal{D}_q^{(m)}(\tilde{\gamma}_1) = e^{iq \int_{\tilde{\gamma}_1} \tilde{A}_1} \tag{2.22}$$

We will discuss soon why the defects take this form. The electric defect, in this example, is called a **Wilson Line**, and the magnetic defect an **'t Hooft Line**. We have already seen generally how our symmetry operators will act on these defects from Equation 2.12, so we won't restate this, but note that $\mathcal{D}_q^{(e)}(\gamma_1)$ is charged only under the electric symmetry, and the same for the magnetic defect and the magnetic symmetry.

The physical interpretation of these defects are 'infinitely massive'[8] electric and magnetic monopoles with worldline given by the line γ_1 that defines them [10]. So, keeping in mind that these are actual physical objects in the theory, we should insert the defects into the path integral of the theory to see how they may affect the theory

$$\langle \mathcal{D}_q^{(e)}(\gamma_1)\rangle = \int DA_1 D\tilde{A}_1 e^{i\int_{\mathcal{M}_4} F_2\wedge * F_2} e^{iq\int_{\gamma_1} A_1} \tag{2.23}$$

$$= \int DA_1 D\tilde{A}_1 e^{i\int_{\mathcal{M}_4} F_2\wedge * F_2 + qA_1\wedge\delta^{(3)}(\gamma_1)} \tag{2.24}$$

such that the Equation of motion for A_1 becomes

$$d * F_2 = q\delta^{(3)}(\gamma_1) \tag{2.25}$$

and similar for inserting the 't Hooft line in the path integral. Therefore, we see that the equations of motion are actually affected by the presence of these defects in the theory. This equation of motion also serves as another way to derive the action of the symmetry operator on the defect, in a similar way to how we obtained this from the Ward identity.

We know that these symmetries must be $U(1)$ due to the symmetry arising from the shift $A_1 \to A_1 + \lambda_1$. This then means that λ_1 must be U(1)-valued. We can perhaps show this more explicitly

$$\mathcal{D}_q^{(e)}(\gamma_1) \to e^{iq\int_{\gamma_1} A_1+\lambda_1} \tag{2.26}$$

$$= e^{iq\int_{\gamma_1}\lambda_1}\mathcal{D}_q^{(e)}(\gamma_1) \tag{2.27}$$

$$\overset{!}{=} e^{iq\alpha Link(\Sigma_2,\gamma_1)}\mathcal{D}_q^{(e)}(\gamma_1) \tag{2.28}$$

$$\Rightarrow \alpha Link(\Sigma_2,\gamma_1) = \int_{\gamma_1}\lambda_1 \tag{2.29}$$

where α is a $U(1)$ parameter, i.e. $\alpha \in U(1)$ and the linking number is just an integer, so $\alpha Link(\Sigma_2,\gamma_1) \in U(1)$ also. Therefore,

$$\int_{\gamma_1}\lambda_1 \in U(1) \tag{2.30}$$

Not only does this tell us that the symmetry is in fact $U(1)$, it also demonstrates why we picked the form of $\mathcal{D}_q^{(e)}(\gamma_1)$ - this defect transforms in the correct way under the 1-form symmetry, and so this is the suitably charged operator of the symmetry. The same argument also applies to the magnetic defect, and so the symmetry group of this Maxwell theory is $G^{(1)} = U(1)\times U(1)$.

[8] A better way of thinking about this might be that their masses are very heavy compared to everything else in the theory.

Defect Group

We are now finished with our example, which has allowed us to demonstrate many of the concepts of higher-form symmetries. There are just a few more properties of higher-form symmetries that we wish to discuss before moving on.

Suppose we have a p-form symmetry group $G^{(p)}$, which we know to be abelian. We would like to know what the possible charges are for the defects, i.e. what values of q can the defects have. Define the **Pontryagin Dual Group** of $G^{(p)}$ as [10]

$$\widehat{G}^{(p)} = \{\phi : G^{(p)} \to U(1) \mid \phi \text{ a homomorphism}\} \tag{2.31}$$

In the language of homological algebra, this would be written as

$$\widehat{G}^{(p)} = Hom(G^{(p)}, U(1)) \tag{2.32}$$

Consider the example $G^{(p)} = U(1)$: how can we map from $U(1)$ to $U(1)$? We could do nothing, we could square the element, cube it - all will leave us still in $U(1)$. More generally, we could map $g \mapsto g^n$ for $n \in \mathbb{Z}$. So, it turns out that the group of all homomorphisms from $U(1)$ to $U(1)$ correspond to the group of integers under addition [10]

$$\widehat{U(1)} = \mathbb{Z} \tag{2.33}$$

and so if we have the most general form, $G^{(p)} = U(1)^N$, then

$$\widehat{G}^{(p)} = \mathbb{Z}^N \tag{2.34}$$

One might ask how the Pontryagin dual relates to the charges of the defects. Let's view Equation 2.12 in the following way: let $\phi_q : U(1) \to U(1) : e^{i\alpha} \mapsto e^{i\alpha q}$; we can see that this is how the $U(1)$ symmetry operators act on the defects, and so this must mean that for $G^{(p)} = U(1)$, $q \in \mathbb{Z}$. Then, for $G^{(p)} = U(1)^N$, $\widehat{G}^{(p)} = \mathbb{Z}^N$. So for all continuous p-form symmetries, the corresponding defects have integer charge. Note that this is the case for *continuous* p-form symmetries - we will see how this changes for discrete p-form symmetries in the next section.

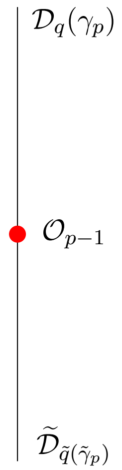
We can put our argument for the Pontryagin dual in reverse also, and consider, for Pontryagin dual group $\widehat{G}^{(p)}$, what is the corresponding $G^{(p)}$? It turns out that [10]

$$\widehat{\widehat{G}^{(p)}} = G^{(p)} \tag{2.35}$$

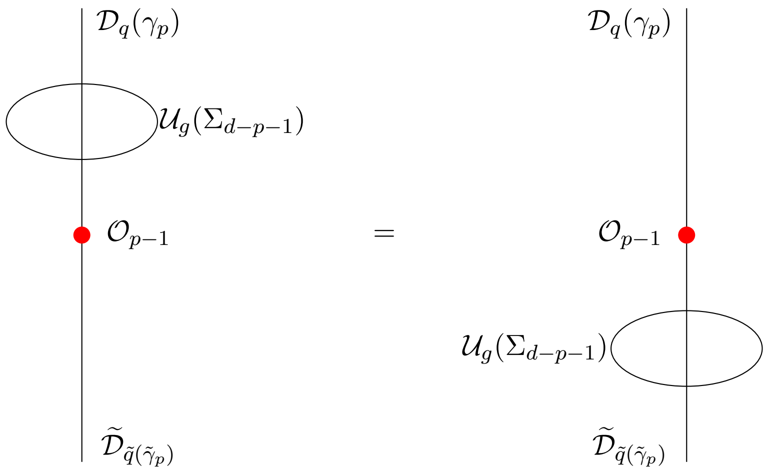
which is easy to see by considering the maps $\phi_\alpha : \mathbb{Z} \to U(1)$ instead, for $\alpha \in U(1)$. Thus, when discussing higher-form symmetries we can talk about the symmetry group $G^{(p)}$ or the **defect group** $\widehat{G}^{(p)}$ interchangeably, as is often done in the literature. See [43] for more on this correspondence.

So far, we have only considered a single defect charged under a given higher-form symmetry. But as we have seen, the charges of defects form a group, and so we can have more than one defect charged under the symmetry. We want to see how these different defects behave around each other, and how they are related.

Consider two p-dimensional defects, $\mathcal{D}_q(\gamma_p)$, $\widetilde{\mathcal{D}}_{\tilde{q}(\tilde{\gamma}_p)} \in \widehat{G}^{(p)}$. Suppose we have that these defects are connected by some $(p-1)$-dimensional operator $\mathcal{O}_{p-1}$, then we say the defects **screen** each other [10]



and this implies that they have the same charge, from the topological property of the symmetry operators [10]:



such that when we shrink the size of Σ_{d-p-1} to measure the charge of each defect on either side of the equation, we get that $q = \tilde{q}$. Therefore, any defects that screen to each other all have the same charge, and we thus say that [10]

$$\textbf{Screening} \rightarrow \mathcal{D}_q(\gamma_p) \sim \widetilde{\mathcal{D}}_{\tilde{q}(\tilde{\gamma}_p)} \tag{2.36}$$

such that the defect group becomes a group of equivalence classes of screenable defects with the same charge. We call a p-dimensional defect **completely screened** or **endable** if it can be screened to a defect with 0 charge (i.e., the p-dimensional identity operator), and thus this defect has vanishing charge under the symmetry group [42].

We will refer to all the various p-form defect groups of a theory collectively as the defect group, and denote it $\mathbb{D}$, such that

$$\mathbb{D} = \bigoplus_p \mathbb{D}^{(p)} = \bigoplus_p \widehat{G}^{(p)} \tag{2.37}$$

2.2 Discrete Higher-form Symmetries

Now that we have discussed continuous higher-form symmetries, we are in a position to move to discrete higher-form symmetries, which will be a main focus of the remainder of this thesis, especially in our geometric engineering setups. In the previous subsection, all of our higher-form symmetry groups were of the form $G^{(p)} = U(1)^N$, such that these were continuous symmetries. But suppose we picked a $\mathbb{Z}_n$ subgroup of $U(1)$ - this would no longer be continuous, but a discrete p-form symmetry. One might ask if such a symmetry is well-defined mathematically, but Dijkgraaf-Witten theory [20] provides us with the mathematical underpinnings of such a symmetry.

There exist more mathematical introductions to discrete higher-form symmetry, but to begin we wish to give a more intuitive understanding of how they work. First, we mention that for $U(1)$ gauge fields A we have

$$\frac{1}{2\pi}\int dA \in \mathbb{Z} = \widehat{U(1)} \tag{2.38}$$

which is the **Dirac quantization condition** [13]. That is, the electric (or magnetic) flux is quantized by the integers, the Pontryagin dual of the symmetry group. Then, suppose we have a p-form symmetry group $G^{(p)} = \mathbb{Z}_n$ for some gauge field A - what would the corresponding flux be? Well, it turns out that [13]

$$\frac{1}{2\pi}\int dA \in \mathbb{Z}_n = \widehat{\mathbb{Z}}_n \tag{2.39}$$

where we have used that the Pontryagin dual of $\mathbb{Z}_n$ is also $\mathbb{Z}_n$ [10]. This follows quite naturally from our previous discussion of the Pontryagin dual of $U(1)$ - we said that the possible homomorphisms from $U(1)$ to $U(1)$ were represented by the integers, corresponding to squaring, cubing, etc. But now if we have the generator $g \in \mathbb{Z}_n \subset U(1)$ such that $g^n = 1$, then the homomorphism ϕ_n will return to the identity - therefore we can have at most n homomorphisms, corresponding to the elements of $\mathbb{Z}_n$. So, we know that the fields under the discrete p-form symmetry must obey this quantization condition, and that the corresponding charged defects will have charges $k = 0, .., n-1$. Likewise, as the symmetry group is now $\mathbb{Z}_n$, the group parameter of the symmetry operators will now be $\mathbb{Z}_n$ valued, instead of the usual $\alpha \in U(1)$. In the same way we considered $G^{(p)} = U(1)^N$ to be the largest possible continuous p-form symmetry group, we can also see that the largest discrete p-form symmetry group would be [10]

$$G^{(p)} = \prod_i \mathbb{Z}_{n_i} \tag{2.40}$$

for integers n_i, where this product implies the usual direct product of groups.

We have gleaned a considerable understanding of discrete p-form symmetries just from considering the ramifications of the possible symmetry groups and the corresponding Pontryagin duals. Let us now consider the prototypical example of discrete p-form symmetries, BF Theory.

BF Theory

Let A_p, B_{d-p-1} be $U(1)$ gauge fields, with action [13]

$$S = \frac{in}{2\pi} \int_{\mathcal{M}_d} B_{d-p-1} \wedge F_{p+1} \tag{2.41}$$

where $F_{p+1} = dA_p$. The equations of motion are then

$$\frac{n}{2\pi} dA_p = 0, \ \frac{n}{2\pi} dB_{d-p-1} = 0 \tag{2.42}$$

and we can see that the action and the equations of motion are gauge invariant. As these are two $U(1)$ gauge fields, we have that they obey the Dirac quantization condition from above, with integer quantization. Currently, there does not seem to be anything hinting that this theory contains discrete higher-form symmetries - we must consider the partition function for these to appear:

$$\mathcal{Z} = \int [dA][dB] e^{\frac{in}{2\pi} \int B_{d-p-1} \wedge dA_p} \tag{2.43}$$

$$= \int [dA][dB] e^{2\pi i \int \frac{dA_p}{2\pi} \wedge \frac{nB_{d-p-1}}{2\pi}} \tag{2.44}$$

Now we only want to include those A_p whose fluxes are integers, so we sum over these

$$= \sum_{k = \int \frac{dA_p}{2\pi} \in \mathbb{Z}} \int [dA][dB] e^{2kn\pi i \int \frac{B_{d-p-1}}{2\pi}} \tag{2.45}$$

Next, we can use a trick from [13] - the identity

$$\sum_{k \in \mathbb{Z}} \int dx e^{2kn\pi i x} f(x) = \sum_{x \in \mathbb{Z}_n} f(x) \tag{2.46}$$

where in our path integral we take $f(x) = 1, x = \int \frac{B_{d-p-1}}{2\pi}$ such that the path integral becomes

$$\mathcal{Z} = \sum_{\int \frac{B_{d-p-1}}{2\pi} \in \mathbb{Z}_n} \int [dB] \tag{2.47}$$

i.e., only the $\mathbb{Z}_n$-valued gauge fields contribute to the path integral, as we can achieve the same condition for A_p in a similar way. Therefore, the quantum theory actually only allows for $\mathbb{Z}_n$ gauge fields.

We can use an analogy with our $U(1)$ Maxwell theory to consider the defects of the theory:

$$\mathcal{D}_q^{(A)}(\gamma_p) = e^{iq \int_{\gamma_p} A_p} \tag{2.48}$$

$$\mathcal{D}_q^{(B)}(\gamma_{d-p-1}) = e^{iq \int_{\gamma_{d-p-1}} B_{d-p-1}} \tag{2.49}$$

as the Maxwell theory was also a theory of two U(1) gauge fields, and this matches the form of the defects there. The only difference here is that now, $\widehat{G}^{(p)} = \mathbb{Z}_n$, and so the charge $q \in \mathbb{Z}_n$.

In our continuous case, we would have used the equations of motion to obtain conserved currents from which we obtain the symmetry operators. However, we run into an issue here. The 'currents' from the equations of motion are actually just the A and B gauge fields themselves. Also notice that the action S of BF theory contained no Hodge star, and no explicit reference to a metric - this is a **Topological Quantum Field Theory (TQFT)**, as we could put a metric on $\mathcal{M}_d$, call it g, and vary S with respect to g. However, we would see that $\delta_g S = 0$ due to this lack of dependence of the metric. Thus, we would usually define our current from

$$d * j = 0 \tag{2.50}$$

but in the original action we had no metric and so using the Hodge star here is ambiguous. Furthermore, suppose we did introduce some metric, such that the above argument was no longer an issue and we could define the currents as such. We then have, in this theory, that the currents are not gauge invariant, and so these are in fact not really currents at all

$$*j_{d-p} = A_p \to A_p + d\Lambda_{p-1} \neq *j_{d-p} \tag{2.51}$$

where Λ_{p-1} is a gauge parameter. We can show that the 'current' associated to B_{d-p-1} is also not gauge invariant by an identical argument.

While it is not correct to refer to these as currents, they do give us some hints of the existence of discrete higher-form symmetries. Let's make the following transformations

$$A_p \to A_p + \lambda_p \tag{2.52}$$

$$B_{d-p-1} \to B_{d-p-1} + \lambda_{d-p-1} \tag{2.53}$$

where the λ's are $\mathbb{Z}_n$-valued global forms, and see if the action is invariant

$$S \to S + \frac{in}{2\pi}\int \lambda_{d-p-1} \wedge dA_p + B_{d-p-1} \wedge \underbrace{d\lambda_p}_{=0} + \lambda_{d-p-1} \wedge \underbrace{d\lambda_p}_{=0} \tag{2.54}$$

$$= S + \frac{in}{2\pi}\int \underbrace{d\lambda_{d-p-1}}_{=0} \wedge A_p \tag{2.55}$$

$$= S \tag{2.56}$$

So we see that these 'currents' were correct in predicting a $\mathbb{Z}_n^{(p)} \times \mathbb{Z}_n^{(d-p-1)}$ symmetry. As we also have

$$0 = d * j_{d-p} \to d(A_p + \lambda_p) = dA_p = 0 \tag{2.57}$$

and similar for our j_{p+1} current associated to B_{d-p-1}, we have that we can write symmetry operators

$$\mathcal{U}_g^{(A)}(\Sigma_p) = e^{i\alpha \int_{\Sigma_p} A_p} \tag{2.58}$$

$$\mathcal{U}_g^{(B)}(\Sigma_{d-p-1}) = e^{i\alpha \int_{\Sigma_{d-p-1}} B_{d-p-1}} \tag{2.59}$$

where $\alpha \in \mathbb{Z}_n$. Let's put, say, $\mathcal{U}_g^{(A)}(\Sigma_p)$ and $\mathcal{D}_q^{(A)}(\gamma_p)$ side by side:

$$e^{i\alpha \int_{\Sigma_p} A_p}, \; e^{iq \int_{\gamma_p} A_p} \tag{2.60}$$

These look incredibly similar to one another, and one might doubt that the former can actually enact a symmetry transformation on the latter. This is indeed correct. Let's insert $\mathcal{D}_q^{(A)}(\gamma_p)$ into the path integral to get a elucidate which symmetry operators are acting on which defects:

$$\langle \mathcal{D}_q^{(A)}(\gamma_p) \rangle = \int [dA][dB] e^{\frac{in}{2\pi}\int B_{d-p-1} \wedge dA_p} e^{iq \int_{\gamma_p} A_p} \tag{2.61}$$

$$= \int [dA][dB] e^{\frac{in}{2\pi}\int dB_{d-p-1} \wedge A_p + \frac{2\pi q}{n}\delta^{(d-p)}(x \in \gamma_p) \wedge A_p} \tag{2.62}$$

such that we obtain the new equation of motion

$$\frac{dB_{d-p-1}}{2\pi} = \frac{q}{n}\delta^{(d-p)}(x \in \gamma_p) \tag{2.63}$$

We can treat this as a Ward identity similar to Equation 2.7:

$$\frac{dB_{d-p-1}}{2\pi}\mathcal{D}_q^{(A)}(\gamma_p) = \frac{q}{n}\delta^{(d-p)}(x \in \gamma_p)\mathcal{D}_q^{(A)}(\gamma_p) \tag{2.64}$$

and by following the same line of argument as before, we end up with the following

$$e^{i\alpha \int_{\Sigma_{d-p-1}} B_{d-p-1}} \mathcal{D}_q^{(A)}(\gamma_p) = e^{\frac{2\pi i \alpha q Link(\Sigma_{d-p-1}, \gamma_p)}{n}} \mathcal{D}_q^{(A)}(\gamma_p) \tag{2.65}$$

where $\alpha \in \mathbb{Z}_n$ is our usual group parameter. Remarkably, the exponential on the left hand side is exactly $\mathcal{U}_g^{(B)}(\Sigma_{d-p-1})$! So what we have found is that

$$\mathcal{U}_g^{(B)}(\Sigma_{d-p-1})\mathcal{D}_q^{(A)}(\gamma_p) = e^{\frac{2\pi i\alpha q Link(\Sigma,\gamma_p)}{n}}\mathcal{D}_q^{(A)}(\gamma_p) \tag{2.66}$$

and by a similar argument

$$\mathcal{U}_g^{(A)}(\Sigma_p)\mathcal{D}_q^{(B)}(\gamma_{d-p-1}) = e^{\frac{2\pi i\alpha q Link(\Sigma,\gamma_{d-p-1})}{n}}\mathcal{D}_q^{(B)}(\gamma_{d-p-1}) \tag{2.67}$$

i.e. the symmetry operator from the current of A_p actually generates the $\mathbb{Z}_n^{(d-p-1)}$ symmetry, similar for B_{d-p-1} and the $\mathbb{Z}_n^{(p)}$ symmetry. The fact that there is some correspondence between $\mathcal{U}_g^{(A)}$ and $\mathcal{D}_q^{(B)}$ (and vice versa) means that we can notice an interesting fact about the defects of BF theory - they do not necessarily commute. By picking $g = e^{\frac{2\pi i q}{n}}$ such that $\mathcal{U}_g^{(B)}(\Sigma_{d-p-1}) = e^{iq\int_{\Sigma_{d-p-1}} B_{d-p-1}} = \mathcal{D}_q^{(B)}(\Sigma_{d-p-1})$, then we can see this non-commutativity from considering Equation 2.66

$$\mathcal{D}_q^{(B)}(\Sigma_{d-p-1})\mathcal{D}_{\tilde{q}}^{(A)}(\gamma_p) = e^{\frac{2\pi i q\tilde{q} Link(\Sigma,\gamma)}{n}}\mathcal{D}_{\tilde{q}}^{(A)}(\gamma_p)\mathcal{D}_q^{(B)}(\Sigma_{d-p-1}) \tag{2.68}$$

Here we note that in Equation 2.66 the symmetry operator no longer appears on the right hand side - this is a convention that we have chosen; a different convention is used in [13] that would make this more apparent. This non-commutativity of the defects will turn out to be an essential feature of BF theory, as we will see later.

2.3 Global Structure of 4d Gauge Theories

So far, we have only considered higher-form symmetries of abelian gauge theories, and this makes sense as it's somewhat easier to introduce abelian symmetries (as all higher-form symmetries are) into abelian theories. However, non-abelian gauge theories can also enjoy higher-form symmetries. This will be the topic of this subsection, through the lens of different gauge theories with the same Lie algebra. Much of what we discuss in this subsection is simply the authors attempt at making the content of [2] more digestible for those whose knowledge of representations of Lie algebras extends only to a basic understanding of roots and weights.

Consider a theory $\mathcal{T}_{\mathfrak{g}}(G)$, i.e. the gauge theory with gauge group G, and some other theory $\mathcal{T}_{\mathfrak{g}}(\widetilde{G})$, where G and $\widetilde{G}$ have the same Lie algebra $\mathfrak{g}$. To avoid being too abstract, we will think of a theory $\mathcal{T}_{\mathfrak{g}}(G)$ as a partition function. In this thesis, we will mostly concern ourselves with the case $\mathfrak{g} = \mathfrak{su}(n)$, but in general will assume $\mathfrak{g}$ is at least semi-simple. The simple Lie algebras are [10]

$$\mathfrak{g} \in \{\mathfrak{su}(n), \mathfrak{so}(n), \mathfrak{sp}(n), \mathfrak{e}_6, \mathfrak{e}_7, \mathfrak{e}_8, \mathfrak{f}_4, \mathfrak{g}_2\} \tag{2.69}$$

and so the semi-simple Lie algebras are then a direct sum of these. We refer to all the choices of theories, i.e. the theory without a **global structure**, as $\mathcal{T}_{\mathfrak{g}}$, that is, all theories with Lie algebra $\mathfrak{g}$ but without specifying the gauge group G. This is called a **relative field theory** [23] [9].

The correlation functions of local operators of $\mathcal{T}_{\mathfrak{g}}(G)$ and $\mathcal{T}_{\mathfrak{g}}(\widetilde{G})$ are the same for $\mathcal{M}_4 = \mathbb{R}^4$, as these depend only on the Lie algebra, and the correlation functions are also independent of the possible defects [2]. However, for more topologically interesting $\mathcal{M}_4$, this might not be the case. As we wish to consider general $\mathcal{M}_4$, we need to consider how the choices of global structure determine the allowed defect charges, from which we can use Pontryagin duality to determine the higher-form symmetry groups of the theory.

To do this, we follow the arguments of [2] which specialises to $d = 4$ gauge theories (usually with a non-zero theta angle). First, let $\widetilde{G}$ be the universal cover of some group G with Lie algebra $\mathfrak{g}$, where we denote the center of $\widetilde{G}$ as $Z(\widetilde{G})$, such that $G = \widetilde{G}/\Gamma$ for $\Gamma \subseteq Z(\widetilde{G})$. Our Wilson operators must be labelled with a representation of G, such that they are invariant under Γ. That is, given representations R of G such that $\Gamma R = R$ [5], we have that the Wilson lines of the theory are written as [13]

$$\mathcal{W}_q^{(R)}(\gamma_1) = Tr_R\ \mathcal{P}e^{iq\int_{\gamma_1} A_1} \tag{2.70}$$

such that the trace over the representation R ensures gauge invariance of the Wilson line, and $\mathcal{P}$ stands for path-ordering (which we won't discuss). Then, we can use root and weight lattices to consider what the possible spectrum of Wilson (and 't Hooft lines) is for the theory with global structure, as these correspond to the given representations and charges under these representations, respectively.

Given weight lattice Λ of $\mathfrak{g}$, with G having a weight sublattice $\Lambda^G \subset \Lambda$, the charges of the Wilson operators then correspond to points of the lattice $\Lambda^G/W(\mathfrak{g})$ where W is the Weyl group of $\mathfrak{g}$, which is defined as [27]

$$W(\mathfrak{g}) = \langle S_\alpha \rangle,\ S_\alpha : \mathfrak{h} \to \mathfrak{h} : h \mapsto h - 2\frac{\alpha \cdot h}{\alpha \cdot \alpha}\alpha \tag{2.71}$$

where α is a root of $\mathfrak{g}$ and $\mathfrak{h} \subset \mathfrak{g}$ is the Cartan subalgebra, such that the S_α are reflections in the hyperplane orthogonal to α.

We can consider also the charges of the 't Hooft operators. To do this, we must introduce the Langlands-dual (or GNO-dual) Lie algebra of $\mathfrak{g}$, which we refer to as $\mathfrak{g}^*$. To define this, we label the set of roots of $\mathfrak{g}$ as $\Phi(\mathfrak{g})$, and the dual roots defined by [26]

$$\Phi^\vee(\mathfrak{g}) = \{\alpha^\vee = N^{-1}\frac{\alpha}{\alpha \cdot \alpha} \mid \alpha \in \Phi(\mathfrak{g})\} \tag{2.72}$$

[9] There seems to be some differing terminology in the literature around relative field theories. What we are referring to as relative field theories here might be too lax for some people, who might instead refer to this is a 'metatheory', or even just 'a theory with no global structure'. Regardless, we will refer to these theories as relative.

such that $\Phi(\mathfrak{g}^*) = \Phi^\vee(\mathfrak{g})$. In this definition, N is either a number or a diagonal matrix depending on if $\mathfrak{g}$ is simple or semi-simple, respectively. For example, if we have $\mathfrak{g}$ is simple, then $N^2 = \frac{\sum_i 1/(\alpha_i \cdot \alpha_i)}{r}$ where r is the rank of $\mathfrak{g}$, and for the semi-simple case it would be a diagonal matrix of such N for each term in the direct sum of $\mathfrak{g}$. Then, given the corresponding weight lattice for $\mathfrak{g}^*$, call it Λ^*, we have that the 't Hooft operators have charges in $\Lambda^*/W(\mathfrak{g})$ as we have that $W(\mathfrak{g}) = W(\mathfrak{g}^*)$ [2].

Dyonic operators have both electric and magnetic charge, and correspond to the lattice $(\Lambda \times \Lambda^*)/W(\mathfrak{g})$, which contains the electric and magnetic charges from before, as well as the new dyonic charges [2]. Let's refer to these charges as

$$(q_e, q_m) \in (\Lambda \times \Lambda^*)/W(\mathfrak{g}) \tag{2.73}$$

where specifying a global structure G picks which $(q_e, 0)$, $q_e \in \Lambda^G/W(\mathfrak{g})$, will be present in our theory; we decide the allowed q_e by picking all integers that are invariant under the action of Γ, which defines our choice of G. If $(q_e, q_m), (q'_e, q'_m)$ are allowed charges in our theory, then so is $(q_e + q'_e, q_m + q'_m)$ and so the class of allowed charges are $[(q_e, q_m)] \in Z(\widetilde{G}) \times Z(\widetilde{G})$, as identifying charges that can be obtained in this way corresponds to taking the weight lattice modulo the root lattice, which gives the center of $\widetilde{G}$ [2]. We can think of this equivalence as screening the different operators to one another, such that they have the same charge [10]. Note that $\widetilde{G}$ and $\widetilde{G}^*$ have the same center, where $\widetilde{G}^*$ is the universal cover of the Lie groups with Lie algebra $\mathfrak{g}^*$ [2].

The way in which we decide the remaining allowed charges, once we've picked G and thus the allowed $(q_e, 0)$, comes from the dyonic version of the Dirac quantisation condition [2]

$$q_e q'_m - q_m q'_e = 0 \bmod N \tag{2.74}$$

for $Z(G) = \mathbb{Z}_N$ (which we will always assume to be the case in this thesis). Therefore, if we have picked our G such that we know the allowed charge for the Wilson operators, then we can use this condition to find all possible allowed charged operators.

Let's consider the example $\mathfrak{g} = \mathfrak{su}(2)$, which has $Z(SU(2)) = \mathbb{Z}_2$. Then, if we pick $G = \widetilde{G} = SU(2)$, we have $\Gamma = \{1\}$, i.e. the trivial group. Therefore, we have $(1, 0)$ as an allowed charge, and the Dirac quantization condition says

$$q'_m = 0 \bmod 2 \tag{2.75}$$

which means that we would then have charges $(n, 2n) \bmod 2$ for $n \in \mathbb{Z}$ in this theory. This essentially means we would just have the original $(1, 0)$ charge. If instead we picked $\Gamma = \mathbb{Z}_2$, then we have $G = SU(2)/\mathbb{Z}_2 = SO(3)$, and the only choice of $(q_e, 0)$ is $q_e = 0 \bmod 2$. Then, the Dirac quantisation condition says

$$q_m q'_e = 0 \bmod 2 \tag{2.76}$$

Therefore, we could have either $q'_e = 0 \bmod 2, q_m = 1 \bmod 2$, or we could have $q_m = 0 \bmod 2, q'_e = 1 \bmod 2, q'_m = 1 \bmod 2$. These are two possible choices, both with gauge group $SO(3)$. We call the former $SO(3)_+$, and it has charges $(0,1)$, i.e. just a 't Hooft operator. The latter we call $SO(3)_-$, and it has a dyonic operator with charge $(1,1)$. So, if we have Lie algebra $\mathfrak{su}(2)$, we actually end up with 3 possible different global structures, with the $SO(3)_\pm$ theories being related by a 2π shift in theta angle [2].

Let's consider these charged defects of these three separate theories. For $G = SU(2)$, we have a Wilson line with $\mathbb{Z}_2$ electric charge; said differently, we have a $\mathbb{Z}_2^{(1)}$ electric defect group. Considering the arguments of the previous subsection, concerning discrete higher-form symmetries, if we have a Pontryagin dual group $\mathbb{Z}_2^{(1)}$, then this implies we have an 'electric' symmetry group $G^{(1)} = (\mathbb{Z}_2^{(1)})_e$, where the 'e' subscript denotes that this is an electric symmetry. As this symmetry arises from the center of the gauge group, it is sometimes called the **center symmetry**. Following this argument for the remaining global structures, we have that $G = SO(3)_+$ has a $(\mathbb{Z}_2^{(1)})_m$ 'magnetic' 1-form symmetry, and $G = SO(3)_-$ has what is called a 'dyonic' 1-form symmetry $(\mathbb{Z}_2^{(1)})_d$ [10]. Dyonic symmetries will often not be a focus for us, as they require introducing a theta angle, which often we will not discuss.

Therefore, the 'theory' $\mathcal{T}_{\mathfrak{su}(2)}$ has defect group $\mathbb{D} = (\mathbb{Z}_2^{(1)})_e \times (\mathbb{Z}_2^{(1)})_m \times (\mathbb{Z}_2^{(1)})_d$, but $\mathcal{T}_{\mathfrak{su}(2)}(SU(2))$ has only $\mathbb{D} = (\mathbb{Z}_2^{(1)})_e$ etc. That is, the 'theory' defined by only the Lie algebra, which lacks global structure, sees all possible symmetries that could arise, but fixing a global structure gives the explicit symmetries that arise in the theory globally. An example of particular relevance is the standard model. Locally, the standard model has the Lie algebra $\mathfrak{su}(3) \oplus \mathfrak{su}(2) \oplus \mathfrak{u}(1)$[10], and it is usually given that the resulting standard model gauge group is $\widetilde{G} = SU(3) \times SU(2) \times U(1)$, the universal cover. Picking this global structure is of no importance to perturbative physics, yet if we wish to consider line operators in the standard model, the analysis above has shown that we need to be more careful. As discussed in [45], the centre of $\widetilde{G}$ is $\mathbb{Z}_6$, and so the standard model group could actually be any $G = \widetilde{G}/\Gamma$, where $\Gamma \in \{1, \mathbb{Z}_2, \mathbb{Z}_3, \mathbb{Z}_6\}$, with a larger subgroup Γ resulting in a richer magnetic line operator spectrum. It has been pointed out in [5] that the existence of such magnetic line operators can give rise to hadrons and leptons with fractional charge.

[10]The Lie algebra $\mathfrak{u}(1)$ is the trivial Lie algebra generated by the identity matrix.

Chapter **3**

Higher-Form Symmetries from Geometric Engineering

In the previous section, we gave an introduction to both continuous and discrete higher-form symmetries of both abelian and non-abelian QFTs, and showed how we can go about constructing these symmetries from currents which we derived from the Lagrangian. However, not all QFTs admit Lagrangians, and so we are unable to use these methods for such non-Lagrangian theories. We might also consider the question of how to generalise the idea of the global structure of 4d theories given in Section 2.3 to theories in a general dimension. Both of these issues, and more, can be probed by considering QFTs that are geometrically engineered from a string theory[11]. In Section 3.1, we will introduce some of the main ideas of both branes and geometric engineering that we will require for Section 3.2, where we will show how we can analyse the higher-form symmetries that are present in the geometrically engineered QFTs. In Section 3.3 we will show that the higher-form symmetries that we capture in Section 3.2 are defined for theories of the form $\mathcal{T}_{\mathfrak{g}}$, i.e. lacking global structure, and we will show how considering flux non-commutativity allows us to assign global structure to a theory, thus making a choice of which defects appear in the theory.

3.1 Branes and Geometric Engineering

The first aim of this section is to briefly introduce the role of branes in string theories, and how they couple to the p-forms of a given string theory. We assume a knowledge of branes in bosonic string theory, and so all we wish to do in this section is to see how their role generalises to p-form fields, instead of just for strings.

[11]We shall abuse language in this section and refer to M-theory as a string theory, even though it is technically not.

Before we discuss branes, we need to know the spectrum of p-forms present in a given string theory. This can be calculated in a similar way to how one might go about calculating the spectrum of fields for bosonic string theory, except one introduces supersymmetry to the theory by adding a fermionic worldsheet also, such that the worldsheet action is invariant under supersymmetry. Determining the boundary conditions of this fermionic worldsheet corresponds to picking a different spectrum for the theory, and we give a very quick sketch of how this works now, skipping over details and giving just the main ideas.

If we refer to the usual bosonic world sheet as X^μ, then introducing a fermionic world sheet ψ^μ gives world-sheet action [9]

$$S = -\frac{1}{2\pi}\int d^2\sigma \partial_\alpha X^\mu \partial^\alpha X_\mu + \bar{\psi}^\mu \rho^\alpha \partial_\alpha \psi_\mu \tag{3.1}$$

where ρ^α satisfy the 2x2 Clifford algebra. ψ^μ can be split up into it's two components

$$\psi^\mu = \begin{bmatrix} \psi^\mu_- \\ \psi^\mu_+ \end{bmatrix} \tag{3.2}$$

and, for closed fermionic strings, which gives us the Type II theories, we have the following boundary conditions which we can impose for both ψ^μ_+ and ψ^μ_- independently:

- **Ramond (R)**: $\psi_\pm(\sigma+\pi) = \pm\psi_\pm(\sigma)$
- **Neveu-Schwarz (NS)**: $\psi_\pm(\sigma+\pi) = \mp\psi_\pm(\sigma)$

i.e. periodic or antiperiodic [9]. This means we have four sectors: R-R, R-NS, NS-R, NS-NS. Each sector corresponds to strings with a choice of boundary condition for each component, and from these we can learn the spectrum of allowed p-forms. The R-NS and NS-R sectors give fermions, while the R-R and NS-NS sectors give us our bosonic p-forms, and, from considering chirality of ground states, we get the following allowed p-form fields from the R-R sector (the NS-NS sector is the same for the two Type II theories) [9]:

- **IIA**: C_1, C_3
- **IIB**: C_0, C_2, C_4

where the field strength of the 4-form in IIB is self-dual, and both theories have $d = 10$. The Type II theories also have a metric, scalar, and 2-form from the NS-NS sector, but the branes we will consider, called Dp-branes for dimension p, couple to the R-R sector forms [9] and so the other sectors won't be relevant for us. We will be considering just the Type II theories, so we won't mention the spectra of the other three string theories (Type I, $E_8 \times E_8$, $SO(32)$), but we would like to consider M-Theory also. We won't explain why, as this would take a lot of explanation that isn't particularly relevant for us, but the only bosonic forms that M-Theory has is the 3-form A_3, and has d=11.

Now that we have the required bosonic spectra for the string theories we wish to consider, we should introduce the corresponding branes for these theories. Consider the example of Maxwell theory, where we measured the electric charge of Wilson lines by the integral

$$q_e = \frac{1}{2\pi}\int_{\Sigma_2} F_2 \tag{3.3}$$

such that $q_e \in \mathbb{Z}$. We mentioned that the physical interpretation of a Wilson line was the worldline of an infinitely massive electrically charged particle, such that the integral above is actually measuring the electric charge of a point particle. If we also include 't Hooft lines, then the integral

$$q_m = \frac{1}{2\pi}\int_{\Sigma_2} *F_2 \tag{3.4}$$

would measure the associated magnetic charge also. If we considered this point particle to be a 0-brane, then this means in $d = 4$ we have that a 0-brane 'couples' to a 1-form A_1 electrically, and to the 1-form $\widetilde{A}_1$ magnetically (where $*F_2 = d\widetilde{A}_1$). We can generalise this to higher-dimensional branes very analogously; if we measure the electric flux of a p-form by the integral

$$e = \int_{\Sigma_{p+1}} F_{p+1} \tag{3.5}$$

and magnetic flux by

$$m = \int_{\Sigma_{d-p-1}} *F_{p+1} \tag{3.6}$$

then this means we have an electric $D(p-1)$-brane and a magnetic dual $D(d-p-3)$-brane [9]. We can see that this generalises the notion of the electric and magnetic flux of our 1-form gauge field in electromagnetism - if we set $p = 1$, $d = 4$, then we get an electric 0-brane and a magnetic 0-brane, i.e. point particles that give us Wilson and 't Hooft lines. Given the p-form spectra for the string theories we are interested in as above, we then have the following Dp-branes (or M-Branes for M-theory), listed in electric-magnetic dual pairs:

- **IIA**: (D0, D6), (D2, D4)
- **IIB**: (D(-1), D7), (D1, D5), (D3)
- **M-Theory**: (M2, M5)

where the IIB D3-brane is it's own magnetic dual, arising from the self-duality of the field strength of C_4. The $D(-1)$-brane is called a 'D-Instanton' as it is localised in time and space [9]. The last thing we wish to mention about branes is that we can **wrap** branes around k-cycles in our spacetime. Consider the example of the bosonic string on $\mathbb{R}^{25} \times S^1$.

The S^1 of this spacetime is a 1-cycle, and we can have a string wrapped around this cycle. The idea is the same for Dp-branes wrapping k-cycles - the brane itself can wrap around non-trivial cycles of spacetime, and this is what we mean when we say a brane *wraps* a k-cycle.

Now that we have discussed branes, we shall begin to discuss geometric engineering. There is no real fixed definition of geometric engineering, rather it is a collection of tools and ideas that one can use to study QFTs from string theories. To begin, let us say that we have a d-dimensional string theory $\mathcal{S}$ defined on a spacetime $\mathcal{M}_D \times X_{d-D}$. The aim is to define a D-dimensional QFT $\mathcal{T}$ on $\mathcal{M}_D$ that is *not* a quantum gravity theory, i.e. we want gravity to decouple. It is known from string compactifications that [10]

$$G \sim \frac{1}{vol(X_{d-D})} \tag{3.7}$$

where G is the gravitational strength, which is why, usually, we aim to have the extra dimensions small such that the resulting theory is a quantum gravity theory[12]. As we wish to study higher-form symmetries, which are global symmetries, this is not a good sign. There is a no-go theorem that says all global symmetries of a theory of quantum gravity should be gauged, and so if we want to discuss higher-form symmetries without the requirement that we *must* gauge them then this would not be the appropriate context. So, what we wish to do is to decouple gravity by having the extra dimensions be **non-compact**, such that the volume is infinite. Then, our theory $\mathcal{T}$ is not a theory of quantum gravity and we can then look for higher-form symmetries.

The things that define our theory $\mathcal{T}$ are then $\mathcal{S}, X_{d-D}$, and the spacetime it exists on, $\mathcal{M}_D$, and so we can think of a geometric engineering configuration as a dictionary [19]

$$\mathcal{S}_{\mathcal{M}_D \times X_{d-D}} \to \mathcal{T}_{\mathfrak{g},\mathcal{M}_D}(\mathcal{S}, X_{d-D}) \tag{3.8}$$

from a string theory $\mathcal{S}$ to a QFT $\mathcal{T}$, where X_{d-D} is non-compact. We will often just refer to either side of the dictionary as $\mathcal{S}$ and $\mathcal{T}_{\mathfrak{g}}$ when it is clear what the configuration is. We have that if $\mathcal{S}$ is a $d = 10$ string theory, e.g. Type II, then we require X_{d-D} to be Calabi-Yau[13], and we will always assume that $\mathcal{M}_D$ is torsion-free, i.e. there are no torsional cycles on $\mathcal{M}_D$.

Given an space X_{d-D}, we would like to know what the corresponding $\mathcal{T}_{\mathfrak{g}}$ is. All of our examples will involve X_{d-D} being of the form $\mathbb{C}^n/\mathbb{Z}_m$ (or a product of this with a torus), and thus we would like to know what $\mathcal{T}_{\mathfrak{g}}$ these can give. For $\mathbb{C}^2/\Gamma$, with finite $\Gamma \subset SU(2)$, this falls under the McKay correspondence [34], with a table from [19] giving the corresponding $\mathfrak{g}$ for each choice of Γ. For our purposes, we only require that $\Gamma = \mathbb{Z}_N$

[12] As well as answering the question of 'if there are extra dimensions, why can we not see them?'

[13] For M-Theory, if one wishes to obtain a $D = 4$ theory then one can instead have X_7 be a G_2-manifold. We won't discuss these configurations here.

results in $\mathfrak{g} = \mathfrak{su}(N)$. Table 1 in [19] shows how other types of simple Lie algebras might be obtained. A way of thinking about this is that the origin of the orbifold $\mathbb{C}^2/\Gamma$ has a singularity, and the QFT $\mathcal{T}_\mathfrak{g}$ 'lives' at this origin, i.e. $\mathcal{M}_D \times \{0\} \cong \mathcal{M}_D$, which $\mathcal{T}_\mathfrak{g}$ is defined on. One might remember from a bosonic string theory course that branes can give rise to non-abelian gauge theories at the endpoints of strings. We can wrap branes around compact cycles of the orbifold, and as these cycles reach the origin by 'slipping them down' to the origin, they 'vanish', giving us a non-abelian gauge theory at the origin where this QFT lives, depending on the type of singularity, i.e. the choice of Γ [33]. This would then give us a theory $\mathcal{T}_\mathfrak{g}$ defined on $\mathcal{M}_{d-4}$, which would be either a $D = 7$ or $D = 6$ theory depending on if $\mathcal{S}$ was M-theory of a string theory. We can choose $X_{d-D} = T^2 \times \mathbb{C}^2/\Gamma$ to obtain a theory $\mathcal{T}_\mathfrak{g}$ defined on $\mathcal{M}_D$ for $D = 5, 4$ instead. The effect the T^2 has is on the number of supercharges in the resulting SQFT. For instance, for X_4 just the orbifold, if we pick $\mathcal{S} = IIB$, we get the $D = 6$ $\mathcal{N}$=(2,0) theory from [47], but if $\mathcal{S} = IIA$ we get a $D = 6$ $\mathcal{N} = (1, 1)$ theory. In both of these instances, picking instead $X_6 = T^2 \times \mathbb{C}^2/\Gamma$ results in an $\mathcal{N} = 4$ SYM theory with the Lie algebra $\mathfrak{g}$ corresponding to Γ as before [25].

Alternatively, we could consider choosing $X_6 = \mathbb{C}^3/\Gamma$, for finite $\Gamma \subset SU(3)$ instead, as we have done for toric varieties in Section 1.3. In this case, there is not as concrete a classification, but for the E_n theories in [37] there is a correspondence with the toric diagrams of the associated X_6, given that the toric diagram is of the form given by Figure 6 of [16]. An example we will consider later is the E_0 theory, whose toric diagram was given in Section 1.3.

3.2 Higher-form Symmetries from Geometric Engineering

We now move on to studying the higher-form symmetries of the theories we are engineering. There exists a wonderful formula that allows us to extract the defect group of a theory $\mathcal{T}_\mathfrak{g}$ just from the topological data of X_{d-D} in combination with the spectrum of branes of $\mathcal{S}$ [18, 3, 19]

$$\mathbb{D} = \bigoplus_n \mathbb{D}^{(n)} = \bigoplus_{n=p-k+1} \frac{H_k(X_{d-D}, \partial X_{d-D})}{H_k(X_{d-D})} \tag{3.9}$$

where p labels the dimension of the p-branes of $\mathcal{S}$ that we choose to wrap over non-compact k-cycles of X_{d-D}. This formula was first put forward, at least in this form, in [3] after the ideas of [18], in the context of $\mathcal{S}$ being M-Theory, and considering only discrete higher-form symmetries - which we can do by considering only $\text{Tor}\frac{H_k(X_{d-D},\partial X_{d-D})}{H_k(X_{d-D})}$. If we allow ourselves to consider the non-torsional homology as well then we obtain continuous higher-form symmetries, as is done in [19]. We also have from Equation 1.21 that if $H_k(X_6, \mathbb{Z}) = 0$

then we can write the defect group as

$$\mathbb{D}^{(n)} = \bigoplus_{n=p-k+1} H_{k-1}(\partial X_6, \mathbb{Z}) \tag{3.10}$$

where k is the dimension of the non-compact cycles, and these $(k-1)$-cycles in $H_{k-1}(\partial X_6, \mathbb{Z})$ that we are considering are the intersection of the non-compact cycles with the boundary of X_6, such that we obtain a compact $(k-1)$-cycle on the boundary. We cannot do this in all instances, but this greatly simplifies our calculations when we can.

This formula might seem quite miraculous at first, so we will now give some intuition as to why this gives the defect group. We have already discussed in the previous subsection that branes wrapping compact cycles that can then be 'slipped' down to the origin give us non-abelian gauge theories at the origin. So, the compact cycles should not be considered in the above formula, hence why we quotient these out. From [33], we have that the mass of the states corresponding to branes wrapping cycles is given by the volume of the cycle. A physical way to think of this is to think of the energy that we require to wrap this brane around such a cycle - if the cycle is bigger, we would need more energy to stretch this brane around the cycle, and the state will thus have a greater mass. Thus, branes wrapping non-compact cycles - whose volume is infinite - will thus give us infinitely massive states in the theory at the origin. These correspond to our defects, which we said can be thought of as the worldlines of infinitely massive particles. If our non-compact cycle cannot be slipped down to the origin, then this means we cannot screen these defects to the identity operator, and so these are the defects of the non-abelian gauge theory $\mathcal{T}_{\mathfrak{g}}$ (without global structure). Thus, wrapping branes around compact cycles gives us a non-abelian QFT, and wrapping around non-compact cycles gives us defects in the QFT.

We have seen that we can obtain the defect group for various higher-form symmetries using Equation 3.9, but how do we see where these arise in the QFT? By considering Equation 3.10, we know that the non-compact cycles that produce these symmetries will actually be on $\mathcal{M}_D \times \partial X_{d-D}$, and so by using the Künneth formula, as well as the fact that for abelian groups G we have $H^n(M) \otimes G = H^n(M, G)$ [3], we get

$$H^p(\mathcal{M}_D \times \partial X_{d-D}) = \bigoplus_{p=n+m} H^n(\mathcal{M}_D) \otimes H^m(\partial X_{d-D}) = \bigoplus_{p=n+m} H^n(\mathcal{M}_D, G_m) \tag{3.11}$$

where $G_m = H^m(\partial X_{d-D})$. Thus, the p-forms on $\mathcal{M}_D \times \partial X_{d-D}$ then correspond to n-forms in the QFT on $\mathcal{M}_D$. These n-forms are the background gauge fields of the corresponding $(n-1)$-form symmetries that we see in the defect group; we will discuss this in more detail in Section 3.3.

$D = 5$ E_0 SCFT

The first example that we consider is the geometric engineering configuration

$$\text{M-Theory}_{\mathcal{M}_5 \times X_6} \to \mathcal{T}_{\mathcal{M}_5}(\text{M-Theory}, X_6) \tag{3.12}$$

where X_6 is now some (crepant resolution of) orbifold of the form $\mathbb{C}^3/\mathbb{Z}_N$, with $\partial X_6 = S^5/\mathbb{Z}_N$. The example we would like to consider here is when $N = 3$, and the theory that we obtain here is called the E_0 theory, and was introduced in [37] as a non-trivial theory with no symmetry at all, i.e. with trivial gauge group, and it is a non-Lagrangian SCFT. We can compute, however, higher-form symmetries for this theory.

We get the following homology for the boundary by considering Equation 1.15 and the cohomology given in [3]:

$$H_\bullet(S^5/\mathbb{Z}_3) = \{\mathbb{Z}, \mathbb{Z}_3, 0, \mathbb{Z}_3, 0, \mathbb{Z}\} \tag{3.13}$$

From Equation 1.44 we can see that we have $H_k(X_6) = 0$ for $k = 1, 3, 5, 6$, and so we can use Equation 1.20 to construct three short exact sequences, with Equation 1.21 in mind to calculate the defect group:

$$0 \to H_6(X_6) \to H_6(X_6, \partial X_6) \to H_5(\partial X_6) \to 0 \tag{3.14}$$

$$0 \to H_4(X_6) \to H_4(X_6, \partial X_6) \to H_3(\partial X_6) \to 0 \tag{3.15}$$

$$0 \to H_2(X_6) \to H_2(X_6, \partial X_6) \to H_1(\partial X_6) \to 0 \tag{3.16}$$

such that we have three equations of the form of Equation 1.21. In this example, we will include non-torsional homology as well, to see how we obtain continuous higher-form symmetries from geometric engineering also. This means that the top exact sequence is of relevance to us (which it wouldn't be if we were to consider just torsional homology).

As we have $\mathcal{S} = \text{M-Theory}$, the values of p that we can consider are $p = 2, 5$, corresponding to the M2, M5 branes, with the possibility of wrapping these around non-compact $k = 2, 4, 6$ cycles, such that we get, from Equation 3.10

$$\bigoplus_n \mathbb{D}^{(n)} = \bigoplus_{n=p-(k-1)} H_{k-1}(\partial X_6) \tag{3.17}$$

$$= \left(\bigoplus_{n=2-(k-1)} H_{k-1}(\partial X_6) \right)_{M2} \bigoplus \left(\bigoplus_{n=5-(k-1)} H_{k-1}(\partial X_6) \right)_{M5} \tag{3.18}$$

$$= \left(\mathbb{Z}_3^{(1)} \oplus \mathbb{Z}_3^{(-1)} \oplus \mathbb{Z}^{(-3)} \right)_{M2} \oplus \left(\mathbb{Z}_3^{(4)} \oplus \mathbb{Z}_3^{(2)} \oplus \mathbb{Z}^{(0)} \right)_{M5} \tag{3.19}$$

and so we have calculated the defect group for $\mathcal{T}_{\mathcal{M}_5}(\text{M-Theory}, \mathbb{C}^3/\mathbb{Z}_3) \equiv \mathcal{T}_{E_0}$. Again, in the next section we will see that this is the defect group for the relative field theory - we have not yet chosen a global structure.

Something intriguing here is the prediction of a discrete **(-1)-form** and a continuous (-3)-form symmetry. These do not fit into the picture of everything we've discussed so far in this thesis - what would these mean? The existence of the (-1)-form symmetry is acknowledged in [3] and left for future analysis, but the (-3)-form continuous symmetry does not appear as only discrete symmetries are considered. We would like to discuss both of these now.

We have seen that a p-form symmetry is defined by a symmetry operator $\mathcal{U}_g(\Sigma_{d-p-1})$ where Σ_{d-p-1} is a $(d-p-1)$-dimensional surface, that we can deform smoothly without changing it's action on charged defects. We usually associated to this action a constant shift of a p-form in the theory. If we pick $p=-1$, then we see that we obtain a d-dimensional surface Σ_d associated to the symmetry operator. This then means that Σ_d must be the whole spacetime, which raises concerns as to how we can deform the surface without deforming the whole spacetime. There is also the question of what does the symmetry act on; p-form symmetries act on p-dimensional defects, so what (-1)-dimensional objects do we have? At face value, this makes no sense, but if we consider a (-1)-dimensional object to be something with a 0-dimensional worldline, then instantons would be the objects under consideration. A detailed discussion of continuous (-1)-form symmetries can be found in [4, 35], though discrete (-1)-form symmetries seem to be less discussed in the literature.

We then wish to discuss this supposed continuous (-3)-form symmetry. Following the same argument as the (-1)-form symmetry, a (-3)-form symmetry would correspond to a topological (d+2)-dimensional surface Σ_{d+2}. This is clearly not a sensible notion, at least from the viewpoint of a d-dimensional QFT. Even if we considered a (-3)-dimensional object by it's worldline, as we did to make sense of the (-1)-form symmetry, we would get a (-2)-dimensional worldline, which is again not sensible. The issue here is that this symmetry arose from an M2 brane wrapping a non-compact 6-cycle. If we consider the dimensionality of the M2 brane, then we can see that it makes sense for us to wrap the M2 brane around the 2- and 4-cycles, but wrapping around the 6-cycle corresponds to wrapping the brane around the entire X_6 - this is what leads us to this erroneous (-3)-form symmetry, as by dimensionality arguments we can see that this is not possible. Therefore, in Equation 3.9, we must be careful in considering which values of p and k we choose to sum over, i.e. we require $n \geq -1$.

$D = 6$ $\mathcal{N} = (1,1)$ and $\mathcal{N} = (2,0)$ $\mathfrak{su}(N)$ SCFTs

In this next example, we will geometrically engineer the defect group of the $D = 6$ $\mathcal{N} = (2,0)$ $\mathfrak{su}(N)$ SCFT from [47], as well as it's $D = 6$ $\mathcal{N} = (2,0)$ $\mathfrak{su}(N)$ cousin, where the former is obtained from the geometric engineering configuration

$$IIB_{\mathcal{M}_6 \times X_4} \to \mathcal{T}_{\mathfrak{su}(N)}(IIB, X_4) \tag{3.20}$$

where $X_4 = \mathbb{C}^2/\mathbb{Z}_N$, and the latter is obtained by replacing IIB with IIA, as mentioned earlier in this section. We have that

$$\partial X_4 = S^3/\mathbb{Z}_N \tag{3.21}$$

and the homology of $X_4, \partial X_4$ is [3]

$$H_\bullet(X_4) = \{\mathbb{Z}, 0, \mathbb{Z}^{N-1}, 0, 0\} \tag{3.22}$$

$$H_\bullet(S^3/\mathbb{Z}_N) = \{\mathbb{Z}, \mathbb{Z}_N, 0, \mathbb{Z}\} \tag{3.23}$$

We are abusing language by referring to X_4 as $\mathbb{C}^2/\mathbb{Z}_N$ - really we should be referring to it's resolution with a boundary $S^3/\mathbb{Z}_N$ at infinity. However, it is understood that this is what we are referring to as X_4. The fact that $H_{2k+1}(X_4) = 0$ means that we can use Equation 1.21 to simplify our calculations of the defect group. As usual, $H_0(X_4, \partial X_4) = 0$, and so this will not contribute to the defect group, but we get the following short exact sequences

$$0 \to H_4(X_4, \partial X_4) \to H_3(\partial X_4) \to 0 \tag{3.24}$$

$$0 \to H_3(X_4, \partial X_4) \to 0 \tag{3.25}$$

$$0 \to H_2(X_4) \to H_2(X_4, \partial X_4) \to H_1(\partial X_4) \to 0 \tag{3.26}$$

$$0 \to H_1(X_4, \partial X_4) \to H_0(\partial X_4) \to H_0(X_4) \to 0 \tag{3.27}$$

from which we can conclude directly

$$H_4(X_4, \partial X_4) = \mathbb{Z} \tag{3.28}$$

such that we get

$$\frac{H_4(X_4, \partial X_4)}{H_4(X_4)} = \mathbb{Z} \tag{3.29}$$

and we also have

$$\frac{H_2(X_4, \partial X_4)}{H_2(X_4)} = \mathbb{Z}_N \tag{3.30}$$

We can use the Ext functor to get $H_1(X_4, \partial X_4)$ in the following way: We have that $Ext(\mathbb{Z}, H_1(X_4, \partial X_4)) = 0$ from Equation 1.23, and therefore

$$H_0(\partial X_4) = \mathbb{Z} \oplus H_1(X_4 \partial X_4) \overset{3.23}{=} \mathbb{Z} \Rightarrow H_1(X_4 \partial X_4) = 0 \tag{3.31}$$

so then

$$\frac{H_1(X_4, \partial X_4)}{H_1(X_4)} = 0 \tag{3.32}$$

We can also assume by considering the long exact sequence around the degree 3 relative homology that $H_3(X_4, \partial X_4) = 0$. Therefore, our defects will come from wrapping branes around non-compact 2 and 4 cycles, giving discrete and continuous operators in D=6, respectively. For $\mathcal{S} = \text{IIB}$, we then have the following defect group for the D=6 $\mathcal{N} = (2,0)$ $\mathfrak{su}(\text{N})$ theory

$$\mathbb{D} =(\mathbb{Z}^{(3)} \oplus \mathbb{Z}_N^{(2)} \oplus \mathbb{Z}^{(0)})_{D3} \oplus (\mathbb{Z}^{(5)} \oplus \mathbb{Z}_N^{(4)} \oplus \mathbb{Z}^{(2)})_{D5} \tag{3.33}$$

$$\oplus (\mathbb{Z}^{(1)} \oplus \mathbb{Z}_N^{(0)})_{D1} \tag{3.34}$$

$$\oplus (\mathbb{Z}^{(-1)})_{D(-1)} \oplus (\mathbb{Z}_N^{(6)} \oplus \mathbb{Z}^{(4)})_{D7} \tag{3.35}$$

For $\mathcal{S} = \text{IIA}$, we get the defect group for the D=6 $\mathcal{N} = (1,1)$ $\mathfrak{su}(N)$ theory defect group

$$\mathbb{D} =(\mathbb{Z}^{(2)} \oplus \mathbb{Z}_N^{(1)} \oplus \mathbb{Z}^{(-1)})_{D2} \oplus (\mathbb{Z}^{(4)} \oplus \mathbb{Z}_N^{(3)} \oplus \mathbb{Z}^{(1)})_{D4} \tag{3.36}$$

$$\oplus (\mathbb{Z}^{(0)} \oplus \mathbb{Z}_N^{(-1)})_{D0} \oplus (\mathbb{Z}^{(6)} \oplus \mathbb{Z}_N^{(5)} \oplus \mathbb{Z}^{(3)})_{D6} \tag{3.37}$$

D=4 $\mathcal{N} = 4$ Super-Yang-Mills

The main purpose of this example is to illustrate the difficulties that arise when we cannot use Equation 3.10, or when the long exact sequence in Equation 1.20 does not reduce to short exact sequences that allow us to easily read off what the relative homology groups are. We will consider $\mathcal{S} =$ IIA, such that $d = 10$, and we will pick $D = 4$, so we will be considering the geometric engineering

$$\text{IIA}_{\mathcal{M}_4 \times X_6} \to \mathcal{T}_{\mathcal{M}_4}(IIA, X_6) \tag{3.38}$$

We wish to pick X_6 such that we obtain a Super-Yang-Mills (SYM), in particular with a $\mathfrak{g} = \mathfrak{su}(N)$ Lie algebra. It can be shown that the correct geometry to pick is

$$X_6 = T^2 \times \mathbb{C}^2/\mathbb{Z}_N \tag{3.39}$$

We can see that this geometry suggests we are going from IIA to the $D = 6$ $\mathcal{N} = (1,1)$ $\mathfrak{su}(N)$ theory to the D=4 $\mathcal{N} = 4$ $\mathfrak{su}(N)$ SYM, but skipping the stop at the $D = 6$ theory.

We have that the boundary of this geometry is [19]

$$\partial X_6 = T^2 \times S^3/\mathbb{Z}_N \tag{3.40}$$

and we have the homology of the torus is [40]

$$H_\bullet(T^2) = \{\mathbb{Z}, \mathbb{Z}^2, \mathbb{Z}\} \tag{3.41}$$

and so using the Künneth formula we can obtain the homology as follows

$$H_k(T^2 \times S^3/\mathbb{Z}_N) = \bigoplus_{p+q=k} H_p(T^2) \otimes H_q(S^3/\mathbb{Z}_N) \tag{3.42}$$

such that we get [19] [14]

$$H_\bullet(T^2 \times S^3/\mathbb{Z}_N) = \{\mathbb{Z}, \mathbb{Z}^2 \oplus \mathbb{Z}_N, \mathbb{Z} \oplus \mathbb{Z}_N \oplus \mathbb{Z}_N, \mathbb{Z} \oplus \mathbb{Z}_N, \mathbb{Z}^2, \mathbb{Z}\} \tag{3.43}$$

We would like to compute

$$\mathbb{D} = \bigoplus_{n=p-k+1} \frac{H_k(T^2 \times \mathbb{C}^2/\mathbb{Z}_N, T^2 \times S^3/\mathbb{Z}_N)}{H_k(T^2 \times \mathbb{C}^2/\mathbb{Z}_N)} \tag{3.44}$$

but currently we only have the homology of ∂X_6. From now on, we write $X_6 = T^2 \times X_4$, where $X_4 = \mathbb{C}^2/\mathbb{Z}_N$, and $\partial X_4 = S^3/\mathbb{Z}_N$ as in the previous example.

If we have $H_{2k+1}(X_6) = 0$, as in the previous example, we can use Equation 3.10. However, from the Künneth formula, we get

$$H_\bullet(X_6) = H_\bullet(X_4 \times T^2) = \{\mathbb{Z}, \mathbb{Z}^2, \mathbb{Z}^N, \mathbb{Z}^{2N-2}, \mathbb{Z}^{N-1}, 0, 0\} \tag{3.45}$$

which shows us that we cannot in general use Equation 3.10, though we can for non-compact 6-cycles:

$$0 \to H_6(X_6, \partial X_6) \to H_5(\partial X_6) \to 0 \tag{3.46}$$

so therefore

$$\frac{H_6(X_6, \partial X_6)}{H_6(X_6)} = \mathbb{Z} \tag{3.47}$$

This is the extent of what we can gain from the exact sequence in Equation 1.20, and we can wrap our Dp branes of IIA around these non-compact cycles to compute a subgroup of the total defect group from Equation 3.10. The question then remains: if we can't use Equation 3.10, how do we obtain the rest of the defect group? We have already considered $H_\bullet(X_4, \partial X_4)$ in the previous example - then, we can introduce a Künneth formula for relative homology [29]

$$H_k(X \times Y, A \times Y \cup X \times B) = \bigoplus_{k=p+q} (H_p(X, A) \otimes H_q(Y, B)) \oplus Tor(H_p(X, A), H_{q-1}(Y, B)) \tag{3.48}$$

If we then pick $X = X_4, Y = T^2, A = \partial X_4, B = \emptyset$, then we get

$$H_k(X_6, \partial X_6) = \bigoplus_{k=p+q} H_p(X_4, \partial X_4) \otimes H_q(T^2) \tag{3.49}$$

where we have used that $H_q(T^2, \emptyset) = H_q(T^2)$, which follows naturally from the definition of relative homology, as well as the fact that $X \times \emptyset = \emptyset$, and $A \times Y \cup \emptyset = A \times Y$. We also have that as $H_q(T^2)$ is always torsion-free, the Tor factor drops out for all k from the properties of Tor given in Section 1.2. Now, to calculate $H_\bullet(X_6, \partial X_6)$, we just need to calculate $H_\bullet(X_4, \partial X_4)$, which is something we considered in the previous example.

[14] The degree 2 and 3 homology groups here differ from those in [19] - we calculated an extra summand of $\mathbb{Z}$ from the Künneth formula in both.

For $H_2(X_4, \partial X_4)$, however, we have

$$Ext(H_1(\partial X_4), H_2(X_4)) = Ext(\mathbb{Z}_N, \mathbb{Z}^{N-1}) = \frac{\mathbb{Z}^{N-1}}{N\mathbb{Z}^{N-1}} = \mathbb{Z}_N \tag{3.50}$$

so we see that the short exact sequence involving $H_2(X_4, \partial X_4)$ is not necessarily split. To see what $H_2(X_4, \partial X_4)$ is, let's consider the conditions on $H_2(X_6, \partial X_6)$:

$$H_2(X_6, \partial X_6) = \bigoplus_{p+q=2} H_p(X_4, \partial X_4) \otimes H_q(T^2) \tag{3.51}$$

$$= H_2(X_4, \partial X_4) \otimes \mathbb{Z} \tag{3.52}$$

$$= H_2(X_4, \partial X_4) \tag{3.53}$$

i.e. these groups are the same. If we consider the long exact sequence

$$\cdots \to H_2(X_6) \to H_2(X_6, \partial X_6) \to H_1(\partial X_6) \to H_1(X_6) \to \dots \tag{3.54}$$

then we see that the torsional elements in $H_1(\partial X_6)$ get mapped to zero in $H_1(X_6)$, and so when considering just torsional elements we have that this sequence terminates, and thus

$$Tor \frac{H_2(X_6, \partial X_6)}{H_2(X_6)} = \mathbb{Z}_N \tag{3.55}$$

This is what is done in [19], but we cannot use the methods introduced in this thesis to get any further: clearly if we assume that Equation 3.26 splits, i.e. if we pick the zero element of $Ext(H_1(\partial X_4), H_2(X_6))$, then we would have that

$$\frac{H_2(X_6, \partial X_6)}{H_2(X_6)} = \frac{H_2(X_4, \partial X_4)}{H_2(X_6)} = \frac{\mathbb{Z}^{N-1} \oplus \mathbb{Z}_N}{\mathbb{Z}^N} \tag{3.56}$$

which is not a sensible quotient. Therefore, we see that there is greater difficulty in using the long exact sequence from Equation 1.20 when we do not have the condition that $H_{2k+1}(X_6) = 0$ as in the previous two examples. There are other methods we could use to continue this example, but the purpose of this example is more to demonstrate that a variety of techniques in toric geometry, homological algebra, and algebraic topology are sometimes required for more complicated geometric engineering setups.

3.3 Flux Non-Commutativity

In Section 2.3 we introduced the notion of a relative field theory and the concept of global structure, and how choosing a global structure chooses the spectrum of defects in the theory. The defect groups we obtained in the previous section were in fact the defect group for the relative field theory $\mathcal{T}$, and so again we must choose a global structure. This again corresponds to picking only a subgroup of the total defect group, and so we would like to see how we go about picking this global structure from the geometric engineering perspective. This is the aim of this subsection.

Let's suppose that we have obtained a defect group containing the following type of defects

$$\mathbb{D} = (Z^{(p)})_e \oplus (Z^{(q)})_m \tag{3.57}$$

where $(Z^{(p)})_e$ is a Z-valued p-form symmetry obtained from an 'electric' brane, and $(Z^q)_m$ is a q-form symmetry obtained from the magnetically dual brane. Schematically we can then write the electric and magnetic fluxes associated to these symmetries as follows

$$\int_{\Sigma_{p+1}} F_{p+1}, \int_{\Sigma_{q+1}} \widetilde{F}_{q+1} \in Z \tag{3.58}$$

where we are integrating over the field strengths of the $p-$ and q-form symmetries that these correspond to. This is just an analogue of the Dirac quantisation condition. It is these field strengths that appear on $\mathcal{M}_D$ from Equation 3.11, i.e. $F_{p+1} \in H^{p+1}(\mathcal{M}_D), \widetilde{F}_{q+1} \in H^{q+1}(\mathcal{M}_D)$. As these fluxes arise from forms on the boundary of the non-compact X_6, we need to specify boundary conditions for them. Usually, we would just make the natural choice and pick Dirichlet boundary conditions, i.e. have that on the boundary at infinity these forms go to zero. If this were possible, then we would be able to say that our theory has the defect group given as above when we choose a global structure. However, this is not always possible. Let's create unitary flux operators, as is done in [25]

$$\Phi_F = e^{i \int_{\Sigma_{p+1}} F_{p+1}}, \; \Phi_{\widetilde{F}} = e^{i \int_{\Sigma_{q+1}} \widetilde{F}_{q+1}} \tag{3.59}$$

We will treat these operators as acting on states defined on the Hilbert space $\mathcal{H}(\mathcal{M}_D \times \partial X_{d-D})$, and of interest to us is whether we can simultaneously set boundary conditions for the fluxes corresponding to these operators. These ideas were put forth in a more abstract sense in [21], and were considered in for IIB fluxes in [25] and subsequent papers in various contexts such as [3]. We follow the arguments of the latter, with the intention of making it clear how this relates to a restriction of the defect group.

Suppose we have a state $|0\rangle \in \mathcal{H}(\mathcal{M}_D \times \partial X_{d-D})$ such that we have

$$\Phi_F |0\rangle = |0\rangle \tag{3.60}$$

then this would mean that, for this state $|0\rangle$, Φ_F acts as the identity operator, $\mathbf{1}$. Then,

$$\Phi_F = e^{i \int_{\Sigma_{p+1}} F_{p+1}} = \mathbf{1} \;\Rightarrow\; F_{p+1} = 0 \tag{3.61}$$

i.e. by picking such a state $|0\rangle$ we are essentially setting Dirichlet boundary conditions for F_{p+1} on the boundary. By acting with $\langle 0|$ on Equation 3.60, we can thus say that

$$\langle \Phi_F \rangle = 1 \;\leftrightarrow\; \text{Dirichlet boundary for } F_{p+1} \tag{3.62}$$

Now consider the case that

$$\Phi_F \Phi_{\tilde{F}} = e^{2\pi i L(F,\tilde{F})} \Phi_{\tilde{F}} \Phi_F \tag{3.63}$$

i.e. that these two operators do not commute, and their failure to commute is captured by some function $L(F, \tilde{F})$. Then,

$$\begin{aligned}\langle 0| \Phi_F \Phi_{\tilde{F}} |0\rangle &= e^{2\pi i L(F,\tilde{F})} \langle 0| \Phi_{\tilde{F}} \Phi_F |0\rangle &(3.64)\\ &= e^{2\pi i L(F,\tilde{F})} \langle 0| \Phi_{\tilde{F}} |0\rangle &(3.65)\\ &= e^{2\pi i L(F,\tilde{F})} \langle \Phi_{\tilde{F}} \rangle &(3.66)\end{aligned}$$

such that we have

$$\langle \Phi_{\tilde{F}} \rangle = e^{-2\pi i L(F,\tilde{F})} \langle 0| \Phi_F \Phi_{\tilde{F}} |0\rangle \tag{3.67}$$

Therefore, if we let

$$\Phi_{\tilde{F}} |0\rangle = e^{2\pi i L(F,\tilde{F})} |0\rangle \tag{3.68}$$

then we obtain

$$\langle \Phi_{\tilde{F}} \rangle = 1 \iff L(F, \tilde{F}) = 0 \tag{3.69}$$

i.e. we can only give Dirichlet boundary conditions to both $F_{p+1}, \widetilde{F}_{q+1}$ if these flux operators commute. We then pick a state $|0; L\rangle$ such that all $F \in L$ have

$$\langle \Phi_F \rangle = 1 \tag{3.70}$$

i.e. we give Dirichlet boundary conditions to all $F \in L$, and we choose L such that this is a maximal set of fluxes[15]. Doing this then means that the p-form symmetry corresponding to an $F_{p+1} \in L$ is then 'turned off', i.e. we pick a subset of the defect group corresponding to fluxes for all $\tilde{F} \notin L$. This is because if a flux $F \in L$ is zero on the boundary, then it would not contribute to Equation 3.10, and thus we can see that picking boundary conditions corresponds to defining what is called a **polarization**, i.e. a choice of subgroup of the total defect group $\mathbb{D}$. This then takes us from a relative field theory $\mathcal{T}_{\mathcal{M}_D}(\mathcal{S}, X_6)$ to a quantum field theory with global structure $\mathcal{T}_{\mathcal{M}_D}(\mathcal{S}, X_6, L)$.

We can consider the 5d E_0 theory as an example. The torsional fluxes in 5d we obtain for this theory come from $TorH^4(\mathcal{M}_\mathcal{D} \times \partial X_{d-D}), TorH^7(\mathcal{M}_\mathcal{D} \times \partial X_{d-D})$ [3], and we can use Equation 3.11 to confirm the defect group obtained in Equation 3.19

$$\begin{aligned}TorH^4(\mathcal{M}_5 \times \partial X_6) &= \bigoplus_{n+m=4} H^n(\mathcal{M}_5) \otimes TorH^m(S^5/\mathbb{Z}_3) &(3.71)\\ &= (H^0(\mathcal{M}_5) \otimes \mathbb{Z}_3) \oplus (H^2(\mathcal{M}_5) \otimes \mathbb{Z}_3) &(3.72)\\ &= \underbrace{H^0(\mathcal{M}_5, \mathbb{Z}_3)}_{\mathbb{Z}_3^{(-1)}} \oplus \underbrace{H^2(\mathcal{M}_5, \mathbb{Z}_3)}_{\mathbb{Z}_3^{(1)}} &(3.73)\end{aligned}$$

[15] Picking L in such a way corresponds to picking a representation of a Heisenberg algebra, generated by the flux operators - see [21, 38] for more details.

and

$$\begin{aligned} TorH^7(\mathcal{M}_5 \times \partial X_6) &= \bigoplus_{n+m=7} H^n(\mathcal{M}_5) \otimes TorH^m(S^5/\mathbb{Z}_3) && (3.74)\\ &= (H^3(\mathcal{M}_5) \otimes \mathbb{Z}_3) \oplus (H^5(\mathcal{M}_5) \otimes \mathbb{Z}_3) && (3.75)\\ &= \underbrace{H^3(\mathcal{M}_5, \mathbb{Z}_3)}_{\mathbb{Z}_3^{(2)}} \oplus \underbrace{H^5(\mathcal{M}_5, \mathbb{Z}_3)}_{\mathbb{Z}_3^{(4)}} && (3.76) \end{aligned}$$

so we see that we recover the field strengths corresponding to all of the discrete higher-form symmetries in the defect group. Let's pick $F_2 \in H^2(\mathcal{M}_5, \mathbb{Z}_3)$, and $\widetilde{F}_3 \in H^3(\mathcal{M}_5, \mathbb{Z}_3)$. The corresponding flux operators are then

$$\Phi_F = e^{i\int_{\Sigma_2} F_2}, \ \Phi_{\tilde{F}} = e^{i\int_{\Sigma_3} \widetilde{F}_3} \tag{3.77}$$

Then, in a very schematic way we can see that these don't commute, by considering an analogy with the non-commutativity of the defects of BF theory, particularly from Equation 2.68:

$$\begin{aligned} \Phi_F \Phi_{\tilde{F}} &= e^{i\int_{\Sigma_2} F_2} e^{i\int_{\Sigma_3} \widetilde{F}_3} && (3.78)\\ &= e^{i\int_{\partial\Sigma_2=\gamma_1} A_1} e^{i\int_{\partial\Sigma_3=\gamma_2} B_2} && (3.79)\\ &\sim \mathcal{D}_1^{(A)}(\gamma_1)\mathcal{D}_1^{(B)}(\gamma_2) && (3.80)\\ &= e^{\frac{2\pi i \mathrm{Link}(\gamma_2,\gamma_1)}{3}} \mathcal{D}_1^{(B)}(\gamma_2)\mathcal{D}_1^{(A)}(\gamma_1) && (3.81)\\ &\sim e^{\frac{2\pi i \mathrm{Link}(\Sigma_3,\Sigma_2)}{3}} \Phi_{\tilde{F}}\Phi_F && (3.82) \end{aligned}$$

If we write the Linking number as an intersection number, as in Equation 2.11, we get the following expression:

$$\Phi_F \Phi_{\tilde{F}} = e^{\frac{2\pi i}{3}\int_{\mathcal{M}_5} F_2 \wedge \widetilde{F}_3} \Phi_{\tilde{F}}\Phi_F \tag{3.83}$$

which matches the form given in [3].

Therefore we have that in the 5d E_0 theory we need to pick a polarisation of the defect group due to the non-commutativity of the fluxes. This gives an alternative way of considering the global structures of QFTs to [2] that allows us to consider the defect groups of theories in dimensions other than $d = 4$. Interestingly, the E_0 theory was defined to have no gauge group, yet we still had to pick a global structure for the theory. This perhaps shows that having a global gauge group is not the defining feature of having global structure.

Chapter **4**

Anomalies and Global Structures

In this chapter, we would like to illustrate some of the properties and uses of anomalies, particularly with reference to 'bulk-boundary' systems, which we will introduce later on in the chapter. Anomalies are usually associated to the failure of a symmetry to hold under certain changes to the system, e.g. when quantizing the classical theory. In this chapter, we will focus on **'t Hooft anomalies** - the failure of gauging a global symmetry. We will also see how these bulk-boundary systems allow us to glean insight into the origin of global structures of theories.

4.1 SPT Phases

Suppose we have some higher-form symmetry $G^{(p)}$ of a theory $\mathcal{T}$, with a corresponding 'magnetic dual' symmetry $G^{(d-p-2)}$. The presence of a **mixed 't Hooft anomaly** is detected by introducing **background gauge fields** for both of the two symmetries simultaneously, call them B_{p+1}, B_{d-p-1}. These are non-dynamical gauge fields, that transform as $B_{p+1} \to B_{p+1} + d\lambda_p$ for λ_p a $G^{(p)}$ symmetry parameter, similar for B_{d-p-1}. Consider the effect of a gauge transformation of one of these fields on the path integral of $\mathcal{T}$

$$\mathcal{Z}[B_{p+1} + d\lambda_p, B_{d-p-1}] = e^{i\int_{\mathcal{M}_d}\mathcal{A}}\mathcal{Z}[B_{p+1}, B_{d-p-1}] \tag{4.1}$$

We call $\mathcal{A}$ the **anomalous phase**, and this phase is how we detect a mixed 't Hooft anomaly [13]. This anomalous phase is often in the form of a Lagrangian for a TQFT, and shows us that we cannot gauge both symmetries at the same time. While anomalies do not necessarily indicate that our theory is somehow *bad*, we would like to see how we can obtain a theory free of anomalies.

We can write the anomalous phase $\mathcal{A}$ as a d-dimensional form

$$\mathcal{A} \equiv \mathcal{A}[\lambda_p, B^{(m)}_{d-p-1}] \tag{4.2}$$

Something we can do to cancel this anomaly is to define the **Symmetry Protected Topological (SPT) phase** $\widehat{\mathcal{A}}$ as follows [13]

$$\widehat{\mathcal{A}}[B_{p+1}+d\lambda_p, B_{d-p-1}] = \widehat{\mathcal{A}}[B_{p+1}, B_{d-p-1}] + d\mathcal{A}[\lambda_p, B_{d-p-1}] \tag{4.3}$$

Notice that the SPT phase is a $(d+1)$-form functional of our background gauge fields, such that it's transformation under the gauge symmetry is a total derivative of the anomalous phase. For us to introduce this SPT phase to our theory, we must then define a $(d+1)$-dimensional manifold to integrate this form over. There is a natural choice here, namely a manifold $\mathcal{N}_{d+1}$ such that $\partial\mathcal{N}_{d+1} = \mathcal{M}_d$[16]. We can then introduce an anomaly-free path integral in the following way

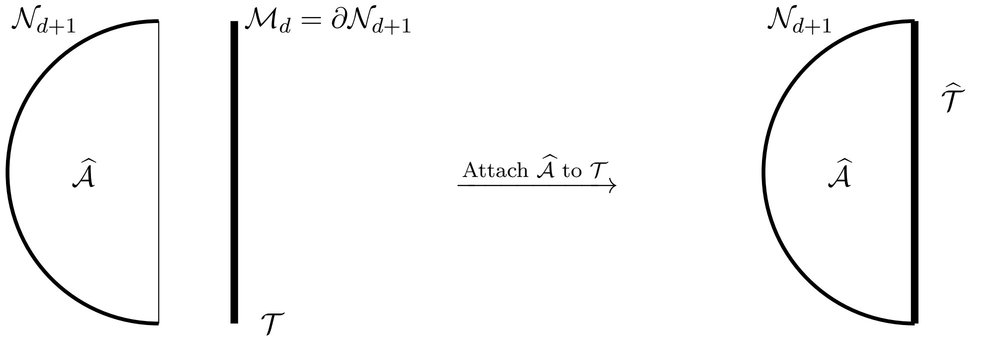
$$\widehat{\mathcal{Z}}[B_{p+1}, B_{d-p-1}] = e^{-i\int_{\mathcal{N}_{d+1}} \widehat{\mathcal{A}}}\mathcal{Z}[B_{p+1}, B_{d-p-1}] \tag{4.4}$$

Then, we have that

$$\begin{aligned}
\widehat{\mathcal{Z}}[B^{\lambda}_{p+1}, B_{d-p-1}] &= e^{-i\int_{\mathcal{N}_{d+1}} \widehat{\mathcal{A}}[B^{\lambda}_{p+1},]}\mathcal{Z}[B^{\lambda}_{p+1},] &(4.5)\\
&= e^{-i\int_{\mathcal{N}_{d+1}} \widehat{\mathcal{A}}[B_{p+1},]+d\mathcal{A}}\, e^{i\int_{\mathcal{M}_d} \mathcal{A}}\mathcal{Z}[B_{p+1},] &(4.6)\\
&= e^{-i\int_{\mathcal{N}_{d+1}} \widehat{\mathcal{A}}[B_{p+1},]}\, e^{-i\int_{\partial\mathcal{N}_{d+1}=\mathcal{M}_d} \mathcal{A}}\, e^{i\int_{\mathcal{M}_d} \mathcal{A}}\, \mathcal{Z}[B_{p+1},] &(4.7)\\
&= e^{-i\int_{\mathcal{N}_{d+1}} \widehat{\mathcal{A}}[B_{p+1},]}\mathcal{Z}[B_{p+1},] &(4.8)\\
&= \widehat{\mathcal{Z}}[B_{p+1}, B_{d-p-1}] &(4.9)
\end{aligned}$$

where $B^{\lambda}_{p+1} = B_{p+1} + d\lambda_p$, and we've omitted B_{d-p-1} in our working for brevity. We can thus see that including this SPT phase in the path integral of a theory $\mathcal{T}$ gives us an anomaly-free theory $\widehat{\mathcal{T}}$. This tells us that the anomaly is a manifestation of a boundary term of a theory in one dimension higher that we are neglecting to consider. Taking this higher dimensional TQFT into account, we are capturing the full system and thus have no anomaly. The SPT in this case is then sometimes referred to as the **anomaly theory** corresponding to our theory $\mathcal{T}$. There is a common picture for this system, sometimes known as a bulk-boundary system [10]

[16]This idea leads naturally to the study of bordisms, the Atiyah-Segal axioms, and relative field theories - see [36] for a concise introduction to this picture.



Anomalous theory $\mathcal{T}$ and SPT $\widehat{\mathcal{A}}$ Anomaly-free $\widehat{\mathcal{T}}$

This process is referred to as **stacking SPT phases**. We note here that $\partial\mathcal{M}_d = \partial^2\mathcal{N}_{d+1} = 0$, and so this is only possible in cases where our spacetime $\mathcal{M}_d$ has no boundary.

It will be illustrative to see some examples. We will consider both the 4d Maxwell theory, and BF theory in general dimension.

Maxwell Theory

Usually when we have a global symmetry, we try to see if we can turn this into a gauge symmetry by letting our symmetry depend on spacetime. We will see what happens when we do this for our electric and magnetic 1-form symmetry. As with gauging a 0-form symmetry by introducing a 1-form gauge field, to gauge a 1-form symmetry we introduce a 2-form background gauge field with transformation [10]

$$B_2^{(e)} \to B_2^{(e)} + d\lambda_1 \tag{4.10}$$

$$B_2^{(m)} \to B_2^{(m)} + d\tilde{\lambda}_1 \tag{4.11}$$

such that now our 1-forms can depend on spacetime, and thus their derivative no longer vanishes in general. In the action, we can couple these 2-form gauge fields to the corresponding current of the global symmetry [10]

$$S = \int F_2 \wedge *F_2 + B_2^{(e)} \wedge *F_2 + B_2^{(m)} \wedge F_2 \tag{4.12}$$

First, let's remove the magnetic 2-form for now and see how the action responds to the electric 2-form gauge transformations, bearing in mind that $F_2 \to F_2 + d\lambda_1$ now:

$$\begin{aligned} S \to & S + \int d\lambda_1 \wedge *F_2 + d\lambda_1 \wedge *F_2 && (4.13)\\ &= S - 2\int \lambda_1 \wedge d * F_2 && (4.14)\\ &= S && (4.15)\end{aligned}$$

as the equations of motion in the presence of this 2-form field are

$$d * F_2 = 0,\ dF_2 = -dB_2^{(e)} \tag{4.16}$$

and so this means this gauge transformation *is* a symmetry. We can clearly do the same thing for the magnetic 2-form gauge field as well. This means that the theory is free of **Pure 't Hooft Anomalies**, i.e. the action is invariant under the gauging of each symmetry individually. However, if we turn on both 2-forms at the same time, and do a gauge transformation of just the electric 2-form, [13]

$$S = \int F_2 \wedge *F_2 + B_2^{(e)} \wedge *F_2 + B_2^{(m)} \wedge F_2 \tag{4.17}$$

$$\overset{(e)}{\to} S + \int d\lambda_1 \wedge *F_2 + d\lambda_1 \wedge *F_2 + B_2^{(m)} \wedge d\lambda_1 \tag{4.18}$$

$$= S - \int 2\lambda_1 \wedge d * F_2 + dB_2^{(m)} \wedge \lambda_1 \tag{4.19}$$

$$= S - \int 2\lambda_1 \wedge (-dB_2^{(m)}) + dB_2^{(m)} \wedge \lambda_1 \tag{4.20}$$

$$= S + \int dB_2^{(m)} \wedge \lambda_1 \tag{4.21}$$

where in Equation 4.20 we used that the equations of motion with both 2-forms turned on are

$$d * F_2 = -dB_2^{(m)},\ dF_2 = -dB_2^{(e)} \tag{4.22}$$

So, we see that we cannot gauge both of these symmetries at once, and this leads to a mixed 't Hooft anomaly. So one can see that if we insist that F_2 is self-dual, i.e. $*F_2 = F_2$, then we have instead that $G^{(1)} = U(1)$, and there would no longer be any mixed 't Hooft anomaly upon gauging.

Let's phrase this in terms of the path integral [10]

$$\mathcal{Z}[B_2^{(e)} + d\lambda_1, B_2^{(m)}] = e^{i\int dB_2^{(m)} \wedge \lambda_1} \mathcal{Z}[B_2^{(e)}, B_2^{(m)}] \tag{4.23}$$

such that we have obtained our anomalous phase

$$\mathcal{A} = dB_2^{(m)} \wedge \lambda_1 \tag{4.24}$$

By following the argument above, this then means we can find the SPT for this anomaly by considering Equation 4.3

$$d\mathcal{A} = dB_2^{(m)} \wedge d\lambda_1 \tag{4.25}$$

such that

$$\widehat{\mathcal{A}}[B_2^{(e)} + d\lambda_1, B_2^{(m)}] = \widehat{\mathcal{A}}[B_2^{(e)}, B_2^{(m)}] + dB_2^{(m)} \wedge d\lambda_1 \tag{4.26}$$

which gives us that

$$\widehat{\mathcal{A}}[B_2^{(e)}, B_2^{(m)}] = dB_2^{(m)} \wedge B_2^{(e)} \tag{4.27}$$

Therefore, to achieve an anomaly-free theory for 4d Maxwell, we can consider the following path integral:

$$\widehat{\mathcal{Z}}[B_2^{(e)}, B_2^{(m)}] = e^{-i\int_{\mathcal{N}_5} B_2^{(e)}\wedge dB_2^{(m)}}\mathcal{Z}[B_2^{(e)}, B_2^{(m)}] \tag{4.28}$$

Something worth mentioning is that this phase is of the form of a BF theory with $N = 1$, which will become relevant later on.

BF Theory

For BF theory, we have a $\mathbb{Z}_N^{(p)} \times \mathbb{Z}_N^{(d-p-1)}$ global symmetry, and we wish to gauge this to find an 't Hooft anomaly. Note that these symmetries are not electromagnetically dual. To do this, we introduce a pair of background gauge fields $C_{p+1}, \widetilde{C}_{d-p}$ such that

$$C_{p+1} \to C_{p+1} + d\lambda_p \tag{4.29}$$

$$\widetilde{C}_{d-p} \to \widetilde{C}_{d-p} + d\widetilde{\lambda}_{d-p-1} \tag{4.30}$$

and couple these to the 'currents' of the symmetry [13]

$$S = \frac{iN}{2\pi}\int_{\mathcal{M}_d} B_{d-p-1}\wedge dA_p - B_{d-p-1}\wedge C_{p+1} - \widetilde{C}_{d-p}\wedge A_p \tag{4.31}$$

The equations of motion with these gauge fields included are modified:

$$dA_p = -\widetilde{C}_{d-p} \tag{4.32}$$

$$dB_{d-p-1} = -C_{p+1} \tag{4.33}$$

which we can then use to see that $dC_{p+1} = 0$ and $d\widetilde{C}_{d-p} = 0$.

Then, we can see there is a mixed 't Hooft anomaly between these gauge symmetries, by considering a $\mathbb{Z}_N^{(p)}$ gauge transformation:

$$\delta S = \frac{iN}{2\pi}\int B_{d-p-1}\wedge d\lambda_p - B_{d-p-1}\wedge d\lambda_p - \widetilde{C}_{d-p}\wedge \lambda_p \tag{4.34}$$

$$= \frac{-iN}{2\pi}\int \widetilde{C}_{d-p}\wedge\lambda_p \tag{4.35}$$

such that we get an anomalous phase

$$\mathcal{A}[\lambda_p, \widetilde{C}_{d-p}] = \frac{-iN}{2\pi}\widetilde{C}_{d-p}\wedge\lambda_p \tag{4.36}$$

Then, we can derive the SPT phase for this anomaly in the same way as before:

$$d\mathcal{A} = \frac{-iN}{2\pi}\widetilde{C}_{d-p}\wedge d\lambda_p \tag{4.37}$$

such that

$$\widehat{\mathcal{A}}[C_{p+1}, \widetilde{C}_{d-p}] = \frac{-iN}{2\pi}\widetilde{C}_{d-p}\wedge C_{p+1} \tag{4.38}$$

and therefore our anomaly-free theory is

$$\widehat{\mathcal{Z}}[C_{p+1}, \widetilde{C}_{d-p}] = e^{\frac{iN}{2\pi}\int_{\mathcal{N}_{d+1}}\widetilde{C}_{d-p}\wedge C_{p+1}}\mathcal{Z}[C_{p+1}, \widetilde{C}_{d-p}] \tag{4.39}$$

Gauged Theories

We can see from both of our examples, as well as by considering the definition in Equation 4.3, that our SPT phases also have an 'anomaly' from the total derivative that we use to cancel the anomaly of our QFT. Suppose we had an SPT phase defined on some $\mathcal{N}_{d+1}$ such that $\partial\mathcal{N}_{d+1} = 0$. Then we would have [10]

$$\int_{\mathcal{N}_{d+1}} d\mathcal{A}[\lambda_p, B_{d-p-1}] = \int_{\partial\mathcal{N}_{d+1}=0} \mathcal{A}[\lambda_p, B_{d-p-1}] = 0 \tag{4.40}$$

Therefore, we can only cancel anomalies using SPT phases defined on manifolds with boundary. By considering the bulk-boundary picture from above, this makes intuitive sense - if the SPT has no boundary then there is nowhere for us to attach the theory $\mathcal{T}$ to achieve the anomaly-free theory.

If we have a introduced background gauge field B_{p+1} for a symmetry $G^{(p)}$ of a theory $\mathcal{T}$ such that we have no 't Hooft anomaly, then we can turn this background gauge field into a **dynamical gauge field** in the following way [10]

$$\mathcal{Z}_{\mathcal{T}/G^{(p)}} = \int [dB_{p+1}]\mathcal{Z}_{\mathcal{T}}[B_{p+1}] \tag{4.41}$$

for continuous $G^{(p)}$, or

$$\mathcal{Z}_{\mathcal{T}/G^{(p)}} = \sum_{[B_{p+1}]} \mathcal{Z}_{\mathcal{T}}[B_{p+1}] \tag{4.42}$$

for discrete $G^{(p)}$, where $[B_{p+1}]$ is the equivalence class of B_{p+1} up to gauge transformations. We have seen this notation for discrete symmetries in Section 2.2 when we considered BF theory. Once we have gauged a background field B_{p+1}, we then write it in lower case, b_{p+1} to easily distinguish between background gauge fields and dynamical gauge fields. We then refer to $\mathcal{T}/G^{(p)}$ as the **gauged theory** [10], and this notation arises from the fact that gauging a symmetry identifies distinct physical states connected by the global symmetry, so it is almost as if the gauge theory is $\mathcal{T}$ modulo this symmetry.

4.2 Anomalies of Non-Abelian Gauge Theories

As well as considering the anomalies of higher-form symmetries in abelian gauge theories as we have done so far, we would also like to understand anomalies of non-abelian gauge theories. We explored the relationship between higher-form symmetries and global structures of 4d non-abelian gauge theories in Section 2.3 through the perspective of representation theory, as well as in higher dimensions in Section 3 by using geometric engineering configurations. We now intend to study the relationship between anomalies and global structure through obstructions to lifting G-bundles to $\widetilde{G}$-bundles, where G is the global structure of our theory, and $\widetilde{G}$ is the universal cover of G. We considered the example of $\widetilde{G} = SU(2)$ and $G = SO(3)$ in Section 2.3, and found that there were two different $SO(3)$ theories, admitting 't Hooft lines and dyonic lines respectively, but to obtain these two different theories from the methods in this section we would have to introduce discrete theta angles, which we will not cover here.

Let's remind ourselves of the main takeaways of Section 2.3. Suppose we have a non-abelian relative gauge theory $\mathcal{T}_{\mathfrak{g}}$ such that the simply-connected group $\widetilde{G}$ corresponding to $\mathfrak{g}$ has center $Z(\widetilde{G})$. Then a theory with global structure $\mathcal{T}_{\mathfrak{g}}(\widetilde{G})$ will have an electric symmetry $G^{(1)} = Z(\widetilde{G})$ [10, 13]. Alternatively, we could pick global structure G=$\widetilde{G}/\Gamma$ where $\Gamma \subseteq Z(\widetilde{G})$, and we have that $\mathcal{T}_{\mathfrak{g}}(G)$ has electric symmetry $G^{(1)} = Z(\widetilde{G})/\Gamma = Z(G)$ and magnetic symmetry $G^{(d-3)} = \widehat{\Gamma}$ [10]. It is mentioned in [13] that this magnetic symmetry of $\mathcal{T}_{\mathfrak{g}}(G)$ is classified by the fundamental group $\pi_1(G)$, but we won't usually refer to the magnetic symmetry in this way.

Before we can discuss the relationship between global structures on non-abelian gauge theories and anomalies, we need to give a gentle introduction to what we mean by G-bundles. In the mathematical literature, one would call a Principal G-bundle the pairing $P = (M, G)$ where M is a smooth manifold and G a Lie group, such that the bundle is a fiber bundle with fibre F the same as the structure group G [40]. Those unfamiliar with fibre bundles need not worry, as we don't intend to give much more explanation that this from a mathematical viewpoint. From our physics perspective, we can consider a G-bundle in the following sense: let M be our spacetime, and G the gauge group that the gauge field(s) of our theory transform under to give a gauge-invariant theory. For example, our 4d Maxwell theory example can be considered a $U(1)$-bundle, where A_1 transforms under $U(1)$ such that the action is gauge invariant. Our field A_1 is actually valued in $U(1)$ [40], and so we write the path integral in the following way

$$\mathcal{Z} = \int [dA_1] e^{iS[A_1]} \tag{4.43}$$

where the integral over A_1 is integrating over $U(1)$-valued fields. If we were to have a different G-bundle, we would be considering a similar integral, just over G instead of $U(1)$.

Then consider the scenario of a G-bundle such that $G \subset \widetilde{G}$ where these are both non-abelian Lie groups of a Lie algebra $\mathfrak{g}$, such that $G = \widetilde{G}/\Gamma$ as above. In our theory of the G-bundle, our gauge field is G-valued, e.g. locally $A_1 = A_1^a T^a$ where T^a are generators of $\mathfrak{g}$ such that the global gauge group is G[17]. For those with knowledge of fibre bundles, this means the transition functions between different patches of M are group elements $g \in G$

[17]Note that locally the gauge field is independent of the global gauge group, or global structure, as was discussed in Section 2.3. This is because A_1 is a connection, not a 1-form, and is thus only defined locally (in a given patch).

such that the A_1 written above in one patch is then given in another patch as [40]

$$A_1 \to g^{-1} A_1 g + g^{-1} dg \tag{4.44}$$

which is just the general form of a gauge transformation of A_1. We would then like to know what stops us from simply summing over all of $\widetilde{G}$, and picking the gauge transformation above to also include those additional Γ elements in $\widetilde{G}$. This is measured by the Γ-valued 2-form w_2, called the **Stiefel-Whitney Class**[18], and is an **obstruction to lifting** a G-bundle to a $\widetilde{G}$-bundle. What we mean by this is that if the class $[w_2] \neq 0$, then the theory will not be invariant under $\widetilde{G}$-valued gauge transformation, or said differently, if $dw_2 = 0$ then we cannot lift from a G-bundle to a $\widetilde{G}$-bundle. A nice explanation of why the obstruction specifically takes the form of a 2-form can be found in [10], but we won't repeat it here. Additionally, physics-friendly explanations of fibre bundles, principal bundles, frame bundles, čech cohomology, and Stiefel-Whitney classes (including exactly how these classes obstruct the lifting of the bundle) can all be found in [40],

We can actually see where the 1-form center symmetry arises by considering this gauge transformation above, by letting $g, g^{-1} \in Z(G)$, such that these g commute with A_1, and letting $g = 1 + \alpha_0 + ...$ such that

$$A_1 \to A_1 + d\alpha_0 \tag{4.45}$$

as usual for an abelian gauge transformation. Then, as we would for Maxwell theory, we can see that

$$F_2 = dA_1 \to d(A_1 + \lambda_1) = F_2 \tag{4.46}$$

for λ_1 a constant $Z(G)$-valued 1-form, such that the field strength invariance implies a global $G^{(1)} = Z(G)$ symmetry. This is why the electric 1-form symmetry is sometimes referred to as the center symmetry - the symmetry arises from the commutativity of the center of the gauge group.

For a theory $\mathcal{T}(G)$, we have a Stiefel-Whitney class w_2 that obstructs us from lifting from a G-bundle to a $\widetilde{G}$-bundle, and if we have introduced a background gauge field B_2 for a global symmetry $G^{(1)}$, then we are attempting to 'divide' out the $G^{(1)}$ factor of the theory by identifying states connected by a symmetry, and so instead of having a G-bundle, we will instead have a $G/G^{(1)}$-bundle, i.e. we only sum over $G/G^{(1)}$-valued dynamical gauge fields in the path integral, instead of G-valued gauge fields. This then means that by turning on a background field B_2^e, we are essentially turning on another Stiefel-Whitney class that obstructs us from lifting the $G/G^{(1)}$-bundle to a G-bundle. If we introduce a magnetic dual background gauge field B_{d-2}^m for $G^{(d-3)}$, then we introduce a coupling to

[18] Some authors will only refer to w_2 as the Stiefel-Whitney class if the groups under consideration are $\widetilde{G} = Spin(N), G = SO(N)$.

the action of the form [10]

$$2\pi \int B^m_{d-2} \wedge w_2 \tag{4.47}$$

We have seen previously that the way we introduce a background gauge field to the theory is by coupling it to the current corresponding to the symmetry. We have already considered that $\mathcal{T}(\widetilde{G})$ has only an electric 1-form symmetry, with no magnetic $(d-3)$-form symmetry. When we consider instead the theory with global structure $\mathcal{T}(G = \widetilde{G}/\Gamma)$, we are introducing the magnetic $(d-3)$-form symmetry at the expense of having a reduced gauge group, where we then obtain the Stiefel-Whitney class w_2 that obstructs us from lifting back to the $\mathcal{T}(\widetilde{G})$ theory with no magnetic symmetry. Therefore, we can consider the Stiefel-Whitney class w_2 to be the conserved current for this symmetry, as it is non-zero iff we have a the $(d-3)$-form symmetry in our theory.

We can see that under gauge transformation, this coupling is not necessarily gauge invariant

$$\to 2\pi \int d\lambda_{d-3} \wedge w_2 = 2\pi \int \lambda_{d-3} \wedge dw_2 \tag{4.48}$$

and so our theory will have an anomaly if $dw_2 \neq 0$. Currently, it does not appear that this is a mixed 't Hooft anomaly, but rather a pure 't Hooft anomaly. We would like to consider now how we can determine if this Stiefel-Whitney class is closed, and we will see that this anomaly is in fact a mixed 't Hooft anomaly if w_2 is not closed.

Let us assume that we have turned on both B^m_{d-2} and B^e_2. Then the bundles we are summing over are $G/G^{(1)}$, with the following obstructions to each lift

$$\begin{array}{c} \widetilde{G} \\ \uparrow\!\!\!\!/ \;\; w_2 \ (\Gamma\text{-valued}) \\ G = \widetilde{G}/\Gamma \\ \uparrow\!\!\!\!/ \;\; B^e_2 \ (G^{(1)}\text{-valued}) \\ G/G^{(1)} \end{array}$$

One might come to the conclusion that the obstruction of a lift from $G/G^{(1)}$ directly to $\widetilde{G}$ is then just some $W_2 = w_2 + B^e_2$, but w_2 is Γ-valued and B^e_2 is $G^{(1)}$-valued, and we do not necessarily have that $\Gamma = G^{(1)}$, and even if we do have this condition, we are lifting from $G/G^{(1)} = (\widetilde{G}/\Gamma)/G^{(1)} \equiv \widetilde{G}/\chi$ to $\widetilde{G}$, where $\chi \subseteq Z(\widetilde{G})$ [10], and so we must be more careful about the obstruction W_2, which will be χ-valued. From the quotient that defined χ, we can see that

$$G^{(1)} = \chi/\Gamma \tag{4.49}$$

and so we have the following short exact sequence

$$0 \to \Gamma \xrightarrow{i} \chi \xrightarrow{\pi} G^{(1)} \to 0 \tag{4.50}$$

where $i : \Gamma \to \chi$ is the inclusion map, and $\pi : \chi \to G^{(1)}$ is the homomorphism that just sends Γ elements to the identity in $G^{(1)}$, and is thus surjective [10]. Therefore, we can use these maps to write W_2 in terms of w_2 and B_2^e such that W_2 is χ-valued [10]

$$W_2 = i(w_2) + \widetilde{B}_2^e \tag{4.51}$$

where $\pi(\widetilde{B}_2^e) = B_2^e$.

For us to have an obstruction to lifting from the $G/G^{(1)}$-bundle directly to the $\widetilde{G}$-bundle, we require $dW_2 = 0$, and so we assume that this is the case. This then means that

$$dW_2 = 0 = i(dw_2) + d\widetilde{B}_2^e \tag{4.52}$$

We are beginning to see the appearance of the dependence of the gauge invariance of the magnetic coupling on the electric background field. We could stop here and leave the anomalous phase in terms of $\widetilde{B}_2^e$, but we don't know all that much about when $\widetilde{B}_2^e$ is closed, so we will discuss this now.

As we have mentioned, $\widetilde{B}_2^e$ is a lift of $G^{(1)}$-valued B_2^e to a χ-valued field. We know that $dB_2^e = 0$, as it is the obstruction to lifting $G/G^{(1)}$-bundles to G-bundles, but this does not necessarily mean that $\widetilde{B}_2^e$ is closed. From the short exact sequence above, we can write the following long exact sequence of cohomology groups [10]

$$\cdots \to H^p(M,\Gamma) \xrightarrow{i} H^p(M,\chi) \xrightarrow{\pi} H^p(M,G^{(1)}) \xrightarrow{\beta} H^{p+1}(M,\Gamma) \to \dots \tag{4.53}$$

where $\beta : H^p(M,G^{(1)}) \to H^{p+1}(M,\Gamma)$ is called the **Bockstein Homomorphism**. This map is analogous to mapping a field to it's field strength; for a $\mathbb{Z}_N$-valued p-form w_p, the Bockstein homomorphism acts as [13]

$$\beta(w_p) = \frac{1}{N} d\widehat{w}_p \bmod N \tag{4.54}$$

where $\widehat{w}_p$ is the 'integral lift' of w_p, i.e. the inclusion of the $\mathbb{Z}_N$-valued $\int w_p$ within the integers: $\int \widehat{w}_p \in \mathbb{Z}$. This way of considering the field strength of discrete gauge fields is what one would do when not using the fact that the path integral of BF theory only sums over $\mathbb{Z}_N$ gauge fields instead of all the $U(1)$ gauge fields considered in the action, as we did in Section 2.2. In [13], the approach we took in Section 2.2 is called a $\mathbb{Z}_N \subset U(1)$ gauge theory; if we wanted to consider purely $\mathbb{Z}_N$ gauge theories, the Bockstein homomorphism is what we would use to define field strengths.

Having introduced the Bockstein homomorphism, we can see that $B_2^e \in H^2(M,G^{(1)})$, and that to map this field to $d\widetilde{B}_2^e \in H^3(M,\chi)$, we can just follow the maps along the exact sequence to $H^3(M,\chi)$, such that we get [10]

$$d\widetilde{B}_2^e = i(\beta(B_2^e)) \tag{4.55}$$

Thus, we can finally write, using Equation 4.52

$$i(dw_2) = -i(\beta(B_2^e)) \tag{4.56}$$

and therefore

$$dw_2 = -\beta(B_2^e) \tag{4.57}$$

so therefore our anomalous phase from the magnetic coupling is

$$\mathcal{A}[B_2^e, \lambda_{d-3}] = -\lambda_{d-3} \wedge \beta(B_2^e) \tag{4.58}$$

such that, when $\beta(B_2^e) \neq 0$, we have a mixed 't Hooft anomaly between B_{d-2}^m and B_2^e. So when does $\beta(B_2^e) = 0$? Well, if the short exact sequence in Equation 4.50 splits, i.e. $\chi = \Gamma \oplus G^{(1)}$, then we have that $\beta(B_2^e) = 0$ and we have no anomaly [10]. We can see this by considering the long exact sequence above by writing the cohomology groups explicitly in terms of their coefficients

$$\cdots \to \Gamma^{b_p} \xrightarrow{i} \chi^{b_p} = \Gamma^{b_p} \oplus (G^{(1)})^{b_p} \xrightarrow{\pi} (G^{(1)})^{b_p} \xrightarrow{\beta} \dots \tag{4.59}$$

and the fact that this sequence is exact means we have that $Im\pi = Ker\beta$. We can see that $Im\pi = (G^{(1)})^{b_p} = Ker\beta$ which thus means, as $B_2^e \in H^2(M, G^{(1)})$, that $\beta(B_2^e) = 0$ if the sequence in Equation 4.50 splits, i.e. if $Ext(G^{(1)}, \Gamma) = 0$.

Now suppose that the sequence doesn't split such that we have anomalous phase $\mathcal{A}$ as given above. Then the corresponding anomaly theory is

$$\widehat{\mathcal{A}}[B_2^e, B_{d-2}^m] = -B_{d-2}^m \wedge \beta(B_2^e) \tag{4.60}$$

From this, we can see that the choice of global structure of a non-abelian gauge theory is connected deeply to the presence of mixed 't Hooft anomalies. There is a similar story for **Higher-Group** Symmetries, but we won't cover these in this thesis.

The Standard Model

For the Standard Model gauge group, $\widetilde{G} = SU(3) \times SU(2) \times U(1)$, we have that $Z(\widetilde{G}) = \mathbb{Z}_6$, and so we have that the possible subgroups Γ are $1, \mathbb{Z}_2, \mathbb{Z}_3, \mathbb{Z}_6$. This then means that for different choices of Γ such that $G = \widetilde{G}/\Gamma$ we have that the electric and magnetic 1-form symmetries are

Γ	$G^{(1)}$	$G^{(d-3)}$
1	$\mathbb{Z}_6$	1
$\mathbb{Z}_2$	$\mathbb{Z}_3$	$\mathbb{Z}_2$
$\mathbb{Z}_3$	$\mathbb{Z}_2$	$\mathbb{Z}_3$
$\mathbb{Z}_6$	1	$\mathbb{Z}_6$

The corresponding exact sequences to Equation 4.50 are

$$0 \to 0 \to \chi \to \mathbb{Z}_6 \to 0 \tag{4.61}$$

$$0 \to \mathbb{Z}_2 \to \chi \to \mathbb{Z}_3 \to 0 \tag{4.62}$$

$$0 \to \mathbb{Z}_3 \to \chi \to \mathbb{Z}_2 \to 0 \tag{4.63}$$

$$0 \to \mathbb{Z}_6 \to \chi \to 0 \to 0 \tag{4.64}$$

For the first and last of these, we have that $\chi = \mathbb{Z}_6$ and the sequences are thus split, and so in these theories there is only one of these background fields to turn on due to only having one symmetry, and thus no mixed 't Hooft anomaly. In principle, we could then go ahead and sum over this discrete background gauge field in the path integral, and have a dynamical 2-form gauge field in the Standard Model. For the second and third sequence, we need to be more careful. If $Ext(G^{(1)}, \Gamma) = 0$, then we have that the sequence splits and there is no mixed 't Hooft anomaly. Using Equation 1.22 we have

$$Ext(\mathbb{Z}_m, \mathbb{Z}_n) = \frac{\mathbb{Z}_n}{m\mathbb{Z}_n} \tag{4.65}$$

where we are interested in calculating this for $m = 2, 3, n = 3, 2$. Considering the first example, we have

$$2\mathbb{Z}_3 = 2\{0, 1, 2\} \text{ mod } 3 \tag{4.66}$$

$$= \{0, 2, 4\} \text{ mod } 3 \tag{4.67}$$

$$= \{0, 2, 1\} \text{ mod } 3 \tag{4.68}$$

$$= \mathbb{Z}_3 \tag{4.69}$$

so therefore $Ext(\mathbb{Z}_2, \mathbb{Z}_3) = 0$, and we can similarly show that $Ext(\mathbb{Z}_3, \mathbb{Z}_2) = 0$, so therefore both of these sequences also split, giving no mixed 't Hooft anomaly. In these two cases, we could have a pair of dynamical 2-form discrete gauge fields in the Standard Model by gauging both of these symmetries.

Anomalies of $\widetilde{G}$=Spin(6)

Perhaps the simplest $\widetilde{G}$ that leads to an 't Hooft anomaly is $\widetilde{G}$ =Spin(6)=SU(4), which has $Z(\widetilde{G}) = \mathbb{Z}_6$, and if we pick $\Gamma = \mathbb{Z}_2$, such that $G = SO(6)$, then we have $G^{(d-3)} = \mathbb{Z}_2$, and $G^{(1)} = \mathbb{Z}_4/\mathbb{Z}_2 = \mathbb{Z}_2$ [10]. Then, putting these into the short exact sequence, we get [10]

$$0 \to \mathbb{Z}_2 \to \mathbb{Z}_4 \to \mathbb{Z}_2 \to 0 \tag{4.70}$$

where the $\mathbb{Z}_4$ in the middle corresponds to the obstruction to lifting from the gauged theory $PSO(6)$=$SO(6)/\mathbb{Z}_2$ to $Spin(6)$. If this short exact sequence splits then we have no 't Hooft anomaly. To check this, we calculate the following

$$Ext(\mathbb{Z}_2, \mathbb{Z}_2) = \frac{\mathbb{Z}_2}{2\mathbb{Z}_2} = \mathbb{Z}_2 \tag{4.71}$$

as clearly $N\mathbb{Z}_N = 0$. Therefore, this sequence does not always split; in fact, $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ is isomorphic to the Klein 4-group, not $\mathbb{Z}_4$ [42] - thus we have a mixed 't Hooft anomaly between the pair of electric and magnetic background fields. This means we cannot sum over these fields to make them dynamical.

4.3 Symmetry TFTs

The content of this subsection ties together nearly everything we have discussed in this thesis, by introducing the Symmetry TFT, or SymTFT for short. This is very much a gentle introduction to SymTFTs, and is more of a 'teaser trailer' for how the ideas we have discussed in this thesis are united in the study of SymTFTs. It is another form of bulk-boundary system, but this time with two boundaries - one on either side of the bulk. The original idea of the SymTFT was proposed in [22], and given it's name in [7], a paper of the supervisor of this thesis. In [7], the SymTFT of a theory $\mathcal{T}$ is constructed by considering the geometric engineering of the theory, but only on the boundary of the geometry X_{d-D} - that is, we have the geometric engineering configurations

$$\mathcal{S}_{\mathcal{M}_D \times X_{d-D}} \to \mathcal{T}_{\mathfrak{g}, \mathcal{M}_D} \tag{4.72}$$

$$\mathcal{S}_{\mathcal{M}_D \times \partial X_{d-D}} \to SymTFT_{D+1} \tag{4.73}$$

Deriving the SymTFT in this way requires introducing differential cohomology, and we would not have enough space to include a sufficient explanation of this here. An overview of differential cohomology for the SymTFT is given in [7], and more comprehensive introductions can be found in [8, 31, 15]. We can, however, still study the SymTFT without discussing it's stringy origin, and thus without needing differential cohomology. We use [42, 10] to do this.

Suppose we have a theory $\mathcal{T}$ with a collection of discrete higher-form symmetries $H = \Pi_i G^{(p_i)}$ such that there is a mixed 't Hooft anomaly between them when we introduce background fields B_{p_i+1}. Then, we can stack an SPT phase $\widehat{\mathcal{A}}$ to the theory, to obtain our anomaly-free theory $\widehat{\mathcal{T}}$. Now that the theory is anomaly-free, we can go ahead and gauge these background fields

$$B_{p_i+1} \to b_{p_i+1} \tag{4.74}$$

such that the b_{p_i+1} are now dynamical gauge fields in the $(d+1, d)$ bulk-boundary theory $\widehat{\mathcal{T}}$, which we cannot do for $\mathcal{T}$ due to the 't Hooft anomaly. We have that the $(d+1)$-

dimensional **action of the SymTFT** is then [42, 10]

$$\xi = \int_{\mathcal{N}_{d+1}} \widehat{\mathcal{A}}[b_{p_1+1}, ..., b_{p_n+1}] + \sum_{i=1}^{n} \frac{n_i}{2\pi} \int b_{p_i+1} \wedge da_{d-p_i-1} \tag{4.75}$$

where the a_{d-p_i-1} are dynamical $\widehat{G}^{(p_i)}$-valued gauge fields in the SymTFT, which are the gauge fields of the dual $G^{(d-p_i-2)}$ symmetry [42]. The n_i are the order of $G^{(p_i)}$, e.g. $G^{(p_i)} = \mathbb{Z}_{n_i}$. This action is composed of the anomaly theory $\widehat{\mathcal{A}}$ for our theory $\widehat{\mathcal{T}}$, and a BF theory - this is the case for higher-form symmetries, but may be different for different kinds of generalised symmetry. Our theory then has two kinds of defects [10, 19]

$$\mathcal{D}_i^b(\Sigma_{p_i+1}) = e^{iq \int_{\Sigma_{p_i+1}} b_{p_i+1}} \tag{4.76}$$

$$\mathcal{D}_i^a(\Sigma_{d-p_i-1}) = e^{i\tilde{q} \int_{\Sigma_{d-p_i-1}} a_{d-p_i-1}} \tag{4.77}$$

where $\Sigma_{p_i+1}, \Sigma_{d-p_i-1}$ are submanifolds within the SymTFT, and $q \in \widehat{G}^{(p_i)}$, $\tilde{q} \in \widehat{\widehat{G}^{(p_i)}} = G^{(p_i)}$.

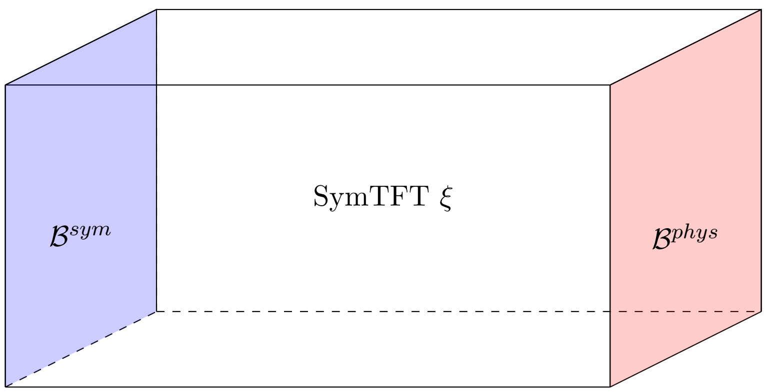
Currently, we have not specified how the SymTFT is a bulk-boundary system - it turns out that the SymTFT actually has two boundaries. We have that one boundary is the original theory $\mathcal{T}$, a relative theory with defect group $\mathbb{D}$ such that not all defects in $\mathbb{D}$ are present in a theory with global structure, $\mathcal{T}(G)$. This is referred to as the **physical, or relative, boundary condition** $\mathcal{B}^{phys}$ [42, 10, 19]. The other boundary, called the **symmetry boundary condition** $\mathcal{B}^{sym}$ is a choice of either Dirichlet or Neumann boundary conditions (b.c.) for each gauge field in the theory [42]. Then, we can say that the $(d+1)$-dimensional manifold $\mathcal{N}_{d+1}$ that the SymTFT is defined on is [19]

$$\mathcal{N}_{d+1} = \mathcal{M}_d \times [-\epsilon, 0] \tag{4.78}$$

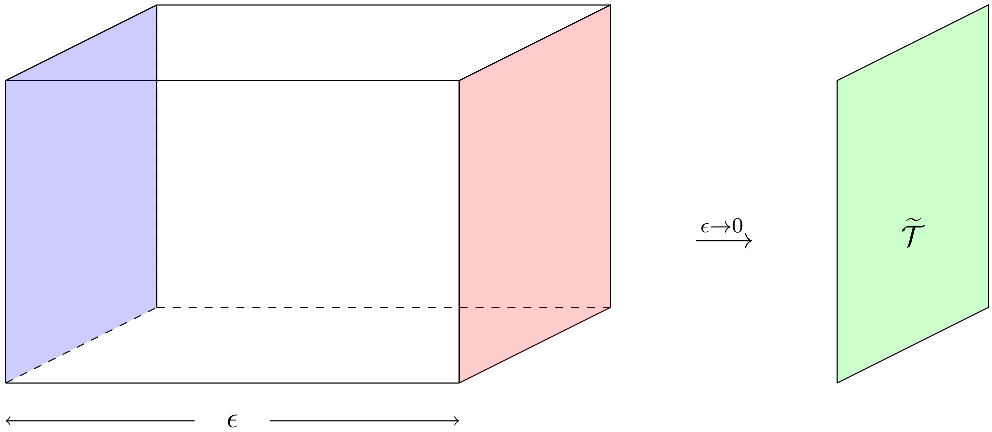
such that

$$\partial\mathcal{N}_{d+1} = \mathcal{M}_d \times \{-\epsilon\} \sqcup \mathcal{M}_d \times \{0\} \tag{4.79}$$

where the former boundary is where $\mathcal{B}^{sym}$ lives and the latter is where $\mathcal{T}$ lives. We can picture this as follows [22, 42, 10]



with the symmetry boundary conditions on one side of the SymTFT ξ and the d-dimensional $\mathcal{T}$ living on the other. As the SymTFT is appropriately named, i.e. it is a topological theory, we can deform it such that we bring the two boundaries together, i.e. send $\epsilon \to 0$. such that we impose the symmetry boundary conditions on the theory $\mathcal{T}$ - this then produces some theory $\widetilde{\mathcal{T}}$ which, depending on the choice of symmetry boundary conditions, can just be the theory $\widehat{\mathcal{T}}$, or a gauged theory $\mathcal{T}/\widetilde{H}$ for $\widetilde{H} \subseteq H$:

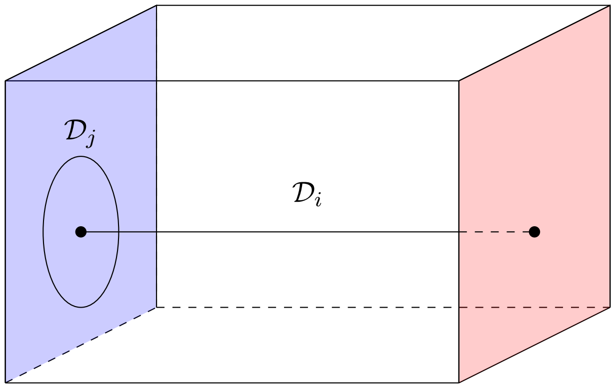


This picture is often referred to as the **sandwich construction**[19]. We may sometimes refer to this informally as a squish, for which we apologise.

Suppose we pick our $\mathcal{B}^{sym}$ to be Dirichlet b.c. for our b_{p_i+1}, such that on the symmetry boundary they take their original background field value B_{p_i+1}, i.e. fixed, non-dynamical fields, and Neumann boundary conditions for the a_{d-p_i-1} fields such that $da_{d-p_i-1} = 0$ on the symmetry boundary. Then, when we squish the boundaries together the BF terms of the SymTFT action will vanish, leaving just the original theory $\widehat{\mathcal{T}}$, i.e. only the SPT phase will remain, with the gauge fields 'ungauged' back to their background fields [10]. One can think of the SymTFT with such a choice of background fields as opening up the theory to see it's component parts.

We have that the choice of $\mathcal{B}^{sym}$ determines what the resulting theory is after we bring the boundaries together. The way this happens is due to where the defects live in the SymTFT. If we pick Dirichlet boundary conditions for some defect $\mathcal{D}_i$, this means that it is a trivial operator on $\mathcal{B}^{sym}$, and so we should have that it ends on $\mathcal{B}^{sym}$ - we can then let the defect end also on $\mathcal{B}^{phys}$. If we pick Neumann b.c. for another defect $\mathcal{D}_j$, then this defect survives in $\mathcal{B}^{sym}$, so we can have this defect 'terminate' in $\mathcal{B}^{sym}$, i.e. it forms a loop in the boundary. We have the following picture for this [10, 42]

[19]There are extensions of the sandwich construction, such as the 'quiche' [22], and 'club sandwich' [11]. See reference 33 of [30] for a greatly anticipated addition to this collection.



where we can thus see that the defects with Neumann b.c. can link those with Dirichlet b.c. such that, if $\mathcal{D}_i$ is G-valued and $\mathcal{D}_j$ is $\widehat{G}$-valued, then these two defects do not commute, i.e. the $\mathcal{D}_j$ is the symmetry operator of the defect $\mathcal{D}_i$ - we saw this 'non-commutativity' of discrete operators both in Equation 2.68 and in Section 3.3. Now, when we squish the two boundaries together, the (p_i+1)-dimensional defect $\mathcal{D}_i$ becomes a p_i-dimensional defect in $\widetilde{\mathcal{T}}$, as it is now in the d-dimensional theory instead of the $(d+1)$-dimensional bulk, and likewise $\mathcal{D}_j$ becomes the p_j-dimensional symmetry operator that measures the charge of this defect[20].

Now suppose that for $H = G^{(p_1)} \times G^{(p_2)}$ we have an 't Hooft anomaly between the two symmetries, such that $\widehat{\mathcal{A}}[b_{p_1+1}, b_{p_2+1}]$ is present in the SymTFT action. Let's consider the anomaly of BF theory, i.e. we have the anomaly theory given in Equation 4.38, with $p_1 = p, p_2 = d-p-1$. Then, we have that the equations of motion for the SymTFT are

$$da_{d-p-1} = \widetilde{c}_{d-p}, \;\; d\widetilde{a}_p = c_{p+1} \tag{4.80}$$

We can see that picking $\mathcal{B}^{sym}$ consistently with these equations means we are only left with Neumann for $a, \widetilde{a}$, and Dirichlet for $c, \widetilde{c}$, which just gives back the anomaly theory after the squish. We cannot pick Neumann conditions for both $c, \widetilde{c}$, which essentially means we *must* 'ungauge' the two fields that exhibit the anomaly - the SymTFT is telling us through it's equations of motion that there is a mixed 't Hooft anomaly between c and $\widetilde{c}$! Therefore, anomalies are detected by the SymTFT as an obstruction to picking $\mathcal{B}^{sym}$ in a way which would leave anomalous fields gauged.

Alternatively, consider having $H = G^{(p)}$ such that $\widehat{\mathcal{A}} = 0$. Then, the action of the SymTFT would just be as in Equation 4.75 without the anomaly theory term. Then, the equations of motion would be

$$da_{d-p-1} = 0 \tag{4.81}$$

We can see that there is no obstruction to setting any boundary conditions here, and can pick Neumann for the b_{p+1} field and Dirichlet for the a_{d-p-1} field. Then, we are choosing

[20] For simplicity, in this thesis we have always assumed that a p-dim symmetry operator acts on a p-dim defect, and so following our conventions we would require that $p_i = p_j$ here. However, it is possible to have $p_j \leq p_i$ [10].

to leave the b_{p+1} field gauged in the resulting d-dim theory, such that $\widetilde{\mathcal{T}} = \mathcal{T}/G^{(p)}$ [10]. We also have that the $G^{(d-p-2)} = \widehat{G}^{(p)}$-valued field a_{d-p-1} becomes a defect $\mathcal{D}^a$ in the gauged theory, with symmetry operator $\mathcal{D}^b$ measuring it's charge - we have a global $G^{(d-p-2)}$ symmetry instead in $\widetilde{\mathcal{T}}$ [10]. This is exactly what was happening in Section 4.2 - consider our example of the Standard Model: if we have $\Gamma = 1$, i.e. we sum over $\widetilde{G}$-bundles, then we have global electric 1-form symmetry $G^{(1)} = \mathbb{Z}_6$. If we then gauge this theory such that we sum over $\widetilde{G}/G^{(1)}$-bundles, giving $\Gamma = \mathbb{Z}_6$, then we obtain a global $G^{(d-3)} = \widehat{\mathbb{Z}}_6 = \mathbb{Z}_6$ dual symmetry in the gauged theory. Essentially, we are seeing that choices of global structure for the relative theory $\mathcal{T}$ correspond to choices of $\mathcal{B}^{sym}$ in the SymTFT.

The SymTFT is clearly a very powerful theory that encapsulates within one paradigm many of the ideas that we have introduced throughout this thesis. If one were to geometrically engineer a relative theory as mentioned at the start of this section, they could also engineer the SymTFT from the boundary geometry - this would allow us study the allowed discrete symmetries, anomalies, and global structures of the theory, all in one simple object. We add that we have discussed only SymTFTs for discrete higher-form symmetries, but the case of continuous higher-form symmetries have been discussed in [6], and non-invertible symmetries in [12].

Chapter 5

Conclusion

In this thesis, we have introduced continuous and discrete higher-form symmetries as a generalisation of our ordinary notion of symmetry, and have seen the different ways in which this allows us to study QFTs beyond what we could do before. The study of higher-form symmetries puts defects in the driver seat for studying QFTs, and these defects have been ubiquitous in our thesis. Something particularly exciting about the importance of electric and magnetic defects when studying higher-form symmetries is that magnetic monopoles would be an example of such a magnetic line defect - a so-far unobserved particle of this kind taking such a prominent role in the study of QFTs with higher-form symmetries is an incredibly exciting prospect, especially with experiments getting closer to detecting these elusive particles [1].

We have also considered global structures of QFTs using both representation theory in Section 2, flux non-commutativity of geometric engineering configurations in Section 3, and through anomalies and the SymTFT in Section 4. This idea is something that a physicist using perturbation theory would not come across, nor expect. It is only until one considers these non-perturbative phenomena, the defects, that one is confronted with the issue. Higher-form symmetries have been pivotal, through their correspondence with defects, in studying QFTs in this way. Perhaps most importantly, recognising that the Standard Model is yet to have it's global structure experimentally verified would just be a technical issue until one begins to consider the higher-form symmetries and the corresponding defects. We also saw the connection between anomalies of higher-form symmetries and the restrictions these impose on the global structure.

The SymTFT allowed us to really bring together all of the topics we studied into one unifying object. Though we only gave a brief look at the power of the SymTFT, it is an object that is rapidly developing at the time of writing, with many gaps in the literature. This made it a particularly exciting closing topic for this thesis, and we look forward to seeing how the SymTFT, and higher-form symmetries in general, progress into the future.

Bibliography

[1] B. Acharya, J. Alexandre, P. Benes, B. Bergmann, S. Bertolucci, A. Bevan, R. Brancaccio, H. Branzas, P. Burian, M. Campbell, S. Cecchini, Y. M. Cho, M. de Montigny, A. D. Roeck, J. R. Ellis, M. Fairbairn, D. Felea, M. Frank, O. Gould, J. Hays, A. M. Hirt, D. L. J. Ho, P. Q. Hung, J. Janecek, M. Kalliokoski, D. H. Lacarrere, C. Leroy, G. Levi, A. Margiotta, R. Maselek, A. Maulik, N. Mauri, N. E. Mavromatos, L. Millward, V. A. Mitsou, E. Musumeci, I. Ostrovskiy, P. P. Ouimet, J. Papavassiliou, L. Patrizii, G. E. Pavalas, J. L. Pinfold, L. A. Popa, V. Popa, M. Pozzato, S. Pospisil, A. Rajantie, R. R. de Austri, Z. Sahnoun, M. Sakellariadou, K. Sakurai, S. Sarkar, G. Semenoff, A. Shaa, G. Sirri, K. Sliwa, R. Soluk, M. Spurio, M. Staelens, M. Suk, M. Tenti, V. Togo, J. A. Tuszynski, A. Upreti, V. Vento, and O. Vives. Moedal search in the cms beam pipe for magnetic monopoles produced via the schwinger effect, 2024.

[2] O. Aharony, N. Seiberg, and Y. Tachikawa. Reading between the lines of four-dimensional gauge theories. *Journal of High Energy Physics*, 2013(8), Aug. 2013.

[3] F. Albertini, M. Del Zotto, I. n. García Etxebarria, and S. S. Hosseini. Higher Form Symmetries and M-theory. *JHEP*, 12:203, 2020.

[4] D. Aloni, E. García-Valdecasas, M. Reece, and M. Suzuki. Spontaneously Broken (-1)-Form U(1) Symmetries. 1 2024.

[5] R. Alonso, D. Dimakou, and M. West. Fractional-charge hadrons and leptons to tell the Standard Model group apart. 4 2024.

[6] A. Antinucci and F. Benini. Anomalies and gauging of U(1) symmetries. 1 2024.

[7] F. Apruzzi, F. Bonetti, I. n. García Etxebarria, S. S. Hosseini, and S. Schafer-Nameki. Symmetry TFTs from String Theory. *Commun. Math. Phys.*, 402(1):895–949, 2023.

[8] C. Bär and C. Becker. *Differential Characters.* Lecture Notes in Mathematics. Springer International Publishing, 2014.

[9] K. Becker, M. Becker, and J. H. Schwarz. *String Theory and M-Theory: A Modern Introduction*. Cambridge University Press, 2006.

[10] L. Bhardwaj, L. E. Bottini, L. Fraser-Taliente, L. Gladden, D. S. W. Gould, A. Platschorre, and H. Tillim. Lectures on generalized symmetries. *Phys. Rept.*, 1051:1–87, 2024.

[11] L. Bhardwaj, L. E. Bottini, D. Pajer, and S. Schafer-Nameki. The club sandwich: Gapless phases and phase transitions with non-invertible symmetries, 2023.

[12] L. Bhardwaj and S. Schafer-Nameki. Generalized Charges, Part II: Non-Invertible Symmetries and the Symmetry TFT. 5 2023.

[13] T. D. Brennan and S. Hong. Introduction to Generalized Global Symmetries in QFT and Particle Physics. 6 2023.

[14] P. Candelas and X. De la Ossa. *Lectures on Complex Manifolds*. Notes de cours du Centre Émile Borel. Institut Henri Poincaré, 2000.

[15] J. Cheeger and J. Simons. Differential characters and geometric invariants. In *Geometry and Topology*, pages 50–80, Berlin, Heidelberg, 1985. Springer Berlin Heidelberg.

[16] C. Closset, M. Del Zotto, and V. Saxena. Five-dimensional SCFTs and gauge theory phases: an M-theory/type IIA perspective. *SciPost Phys.*, 6(5):052, 2019.

[17] D. Cox, J. Little, and H. Schenck. *Toric Varieties*. Graduate studies in mathematics. American Mathematical Society, 2011.

[18] M. Del Zotto, J. J. Heckman, D. S. Park, and T. Rudelius. On the Defect Group of a 6D SCFT. *Lett. Math. Phys.*, 106(6):765–786, 2016.

[19] M. Del Zotto, S. N. Meynet, and R. Moscrop. Remarks on Geometric Engineering, Symmetry TFTs and Anomalies. 2 2024.

[20] R. Dijkgraaf and E. Witten. Topological gauge theories and group cohomology. *Communications in Mathematical Physics*, 129(2):393 – 429, 1990.

[21] D. S. Freed, G. W. Moore, and G. Segal. Heisenberg groups and noncommutative fluxes. *Annals of Physics*, 322(1):236–285, Jan. 2007.

[22] D. S. Freed, G. W. Moore, and C. Teleman. Topological symmetry in quantum field theory, 2023.

[23] D. S. Freed and C. Teleman. Relative quantum field theory, 2014.

[24] D. Gaiotto, A. Kapustin, N. Seiberg, and B. Willett. Generalized Global Symmetries. *JHEP*, 02:172, 2015.

[25] I. n. García Etxebarria, B. Heidenreich, and D. Regalado. IIB flux non-commutativity and the global structure of field theories. *JHEP*, 10:169, 2019.

[26] P. Goddard, J. Nuyts, and D. Olive. Gauge theories and magnetic charge. *Nuclear Physics B*, 125(1):1–28, 1977.

[27] B. Hall. *Lie Groups, Lie Algebras, and Representations: An Elementary Introduction*. Graduate Texts in Mathematics. Springer, 2003.

[28] B. Hassett. *Introduction to Algebraic Geometry*. Cambridge University Press, 2007.

[29] A. Hatcher. *Algebraic topology*. Cambridge University Press, Cambridge, 2002.

[30] J. J. Heckman and M. Hübner. Celestial Topology, Symmetry Theories, and Evidence for a Non-SUSY D3-Brane CFT. 6 2024.

[31] M. J. Hopkins and I. M. Singer. Quadratic functions in geometry, topology,and m-theory, 2005.

[32] K. Hori, S. Katz, A. Klemm, R. Pandharipande, R. Thomas, C. Vafa, R. Vakil, and E. Zaslow. *Mirror symmetry*, volume 1 of *Clay mathematics monographs*. AMS, Providence, USA, 2003.

[33] A. Karch. Field theory dynamics from branes in string theory, 1998.

[34] J. McKay. Graphs, singularities, and finite groups. In *The Santa Cruz Conference on Finite Groups (Univ. California, Santa Cruz, Calif., 1979)*, volume 37 of *Proc. Sympos. Pure Math.*, pages 183–186. Amer. Math. Soc., Providence, RI, 1980.

[35] J. McNamara and C. Vafa. Baby Universes, Holography, and the Swampland. 4 2020.

[36] S. Monnier. A modern point of view on anomalies: Lms/epsrc durham symposium on higher structures in m-theory. *Fortschritte der Physik*, 67(8–9), May 2019.

[37] D. R. Morrison and N. Seiberg. Extremal transitions and five-dimensional supersymmetric field theories. *Nuclear Physics B*, 483(1–2):229–247, Jan. 1997.

[38] D. Mumford. Tata lectures on theta i. 1982.

[39] J. Munkres and J. Munkres. *Elements Of Algebraic Topology*. CRC Press, 2018.

[40] M. Nakahara. *Geometry, topology and physics*. 2003. Bristol, UK: Hilger (1990) 505 p. (Graduate student series in physics).

[41] S. Reffert. The Geometer's Toolkit to String Compactifications. In *Conference on String and M Theory Approaches to Particle Physics and Cosmology*, 6 2007.

[42] S. Schafer-Nameki. Ictp lectures on (non-)invertible generalized symmetries, 2023.

[43] S.-H. Shao. What's done cannot be undone: Tasi lectures on non-invertible symmetries, 2024.

[44] A. Tomasiello. *Geometry of String Theory Compactifications*. Cambridge University Press, 2022.

[45] D. Tong. Line operators in the standard model. *Journal of High Energy Physics*, 2017(7), July 2017.

[46] C. A. Weibel. *An Introduction to Homological Algebra*. Cambridge Studies in Advanced Mathematics. Cambridge University Press, 1994.

[47] E. Witten. Geometric langlands from six dimensions, 2009.